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Random walks on directed graphs and orientations of graphs

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Random walks on directed graphs and orientations of graphs

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The dissertation of Sinan Güven Aksoy is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

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2017

DEDICATION

To Asu, Ercu, Can, and Frankie.

EPIGRAPH

Truth will sooner come out from error
than from confusion.
|Francis Bacon

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Chapter 3 is based in part on the paper Graphs with many strong orientations, SIAM J. Discrete Math., 30(2):1269–1282, 2016, DOI: 10.1137/15M1018885; jointly written with Paul Horn. The dissertation author was the primary author of this work.

In Chapter 4, Section 4.1 is based in part on a unpublished manuscript in preparation, tentatively titled Maximum hitting time of random walks on directed graphs, jointly written with Ran Pan. The dissertation author was the primary author of this work.

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ABSTRACT OF THE DISSERTATION

Random walks on directed graphs and orientations of graphs

by

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We apply spectral theory to study random processes involving directed graphs. In the first half of this thesis, we examine random walks on directed graphs, which is rooted in the study of non-reversible Markov chains. We prove bounds on key spectral invariants which play a role in bounding the rate of convergence of the walk and capture isoperimetric properties of the directed graph. We first focus on the principal ratio, which is the ratio of maximum to minimum values of vertices in the stationary distribution. Improving upon previous bounds, we give a sharp upper bound for this ratio over all strongly connected graphs on vertices. We characterize all graphs achieving the upper bound and give explicit constructions for these extremal graphs. Additionally, we show that under certain conditions, the principal ratio is tightly bounded. We then turn our attention to the first

nontrivial Laplacian eigenvalue of a strongly connected directed graph. We give a lower bound for this eigenvalue, extending an analogous result for undirected graphs to the directed case. Our results on the principal ratio imply this lower bound can be factorially small in the number of vertices, and we give a construction having this eigenvalue factorially small.

In the second half, we apply spectral tools to study orientations of graphs. We focus on counting orientations yielding strongly connected directed graphs, called *strong orientations*. Namely, we show that under a mild spectral and minimum degree condition, a possibly irregular, sparse graph G has "many" strong orientations. More precisely, given a graph G on n vertices, orient each edge in either direction with probability $1/2$ independently. We show that if G satisfies a minimum degree condition of $(1+c_1)\log_2 n$ and has Cheeger constant at least $c_2 \frac{\log_2 \log_2 n}{\log_2 n}$, then the resulting randomly oriented directed graph is strongly connected with high probability. This Cheeger constant bound can be replaced by an analogous spectral condition via the Cheeger inequality. Additionally, we provide an explicit construction to show our minimum degree condition is tight while the Cheeger constant bound is tight up to a $\log_2 \log_2 n$ factor. We conclude by exploring related future work.

Chapter 1

Introduction

1.1 Notation and Preliminaries

We utilize graph theory and matrix analysis notation that is largely standard. A *graph* $G = (V, E)$ is a set of vertices $V := V(G)$ and set of edges $E := E(G)$ where $E \subseteq \{\{u, v\} : u, v \in V\}$. Unless otherwise stated, we assume that G is *simple*, meaning that each edge $\{u, v\} \in E$ consists of a pair of distinct vertices $u, v \in V$, and V is finite. If $\{u, v\} \in E$, we say u and v are *adjacent* and sometimes write $u \sim v$. For each $u \in V(G)$, the *neighborhood* of u, denoted by $N(u) := N_G(u)$ is the set of vertices $\{v : \{u, v\} \in E\}$. The *degree* of a vertex, denoted $d(u)$, is $|N(u)|$. A *walk* of length k is a sequence of vertices $v_0, v_1, \ldots, v_k$ where $\{v_i, v_{i+1}\}$ is an edge. If, for all $u, v \in V$, there exists a walk $u, \ldots, v$, then we say G is *connected*.

A *directed graph* $D = (V, E)$ is defined analogously, except that the edge set $E \subseteq \{(u, v) : u, v \in V\}$ consists of ordered pairs of vertices. That is, a directed edge from vertex u to v is denoted by (u, v) or $u \to v$, and we say v is an out-neighbor of u, or u is an in-neighbor of v. Again, we assume D is simple throughout. For each $u \in V$, the out-neighborhood of u, denoted by $N^+(u)$, is the vertex set $\{v : (u, v) \in E\}$ and the out-degree of u, denoted by $d^+(u)$, is $|N^+(u)|$. Similarly, the in-neighborhood and in-degree of u are denoted by $N^-(u)$ and $d^-(u)$ respectively. A walk of length k is a sequence of vertices $v_0, v_1, \ldots, v_k$ where (v_i, v_{i+1}) is an edge. If, for all $u, v \in V$, there exists walks $u, \ldots, v$ and

$v, \ldots, u$, then we say D is strongly connected.

We will study various matrices associated with graphs. Unless otherwise stated, all matrices are $n \times n$ matrices over the complex numbers $\mathbb{C}$, where n is the number of vertices in the associated graph. We write $\mathbf{1}$ to denote a column vector of ones, I is the identity matrix, and A^T and A^* denote the transpose and conjugate transpose of matrix A, respectively. We say a matrix A is positive and write $A > 0$ if all entries in A are positive. We write eigenvectors as complex-valued functions on the vertex set $V = \{1, \ldots, n\}$; hence $f(u)$ denotes entry u of vector f. For two such complex-valued functions f, g, we let $\langle f, g \rangle = \sum_x f(x)\overline{g(x)}$ denote the usual inner product. We sometimes write $[n]$ to denote $\{1, \ldots, n\}$.

Finally, we will utilize standard asymptotic notation: we say a function $f(n) = O(g(n))$ if for all sufficiently large values of n there exists a positive constant c such that $|f(n)| \le c|g(n)|$; similarly, we write $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$, and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. Lastly, $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ and if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, we write $f(n) \sim g(n)$ or $f(n) = (1 + o(1))g(n)$.

1.2 Random walks on directed graphs

1.2.1 The transition matrix and stationary distribution

We begin by briefly reviewing some relevant concepts concerning random walks on graphs. The study of random walks on graphs is more generally rooted in the study of Markov chains. In what follows, we gear our exposition explicitly towards random walks on directed graphs, although many of the concepts discussed have natural analogs in the undirected case. As we will see, problems that are straightforward for undirected graphs often have relatively complicated counterparts in the directed case. We assume only basic knowledge of probability theory; for a general survey on random walks on graphs, see [34].

A discrete-time, finite, time-homogeneous Markov chain is a sequence of random variables $X_1, X_2, \ldots$ taking values in a finite state space S such that for all t, the sequence satisfies the Markov property and time-homogeneity, i.e.

$$P(X_{t+1} = x_{t+1} \mid X_1 = x_1; \ldots; X_t = x_t) = P(X_{t+1} = x_{t+1} \mid X_t = x_t);$$
$$P(X_{t+1} = x \mid X_t = y) = P(X_t = x \mid X_{t-1} = y):$$

A random walk on a graph $G = (V; E)$ is such a Markov chain $X_0; X_1; \ldots$ defined by a transition probability matrix P in which entry $P(u; v) = P(X_{t+1} = v \mid X_t = u)$ for every $u; v \in V$. Letting $f_t(u) = P(X_t = u)$ denote the probability distribution after t steps, P satisfies

$$f_{t+1} = f_t P;$$

where f_t is viewed as a row vector. Consequently, if f_0 denotes any initial probability distribution, we have

$$f_t = f_0 P^t:$$

Unless otherwise stated, we restrict attention to studying *simple* random walks on a given directed graph $G = (V; E)$, in which the probability transition matrix P is given by

$$P(u; v) = \begin{cases} \frac{1}{d^+(u)}; & \text{if } (u; v) \in E; \\ 0 & \text{otherwise} \end{cases}$$

That is, in a simple random walk, the probability of transitioning from a vertex to any of its out-neighbors is equally likely. More generally, we note that every discrete-time, finite Markov chain can be naturally viewed as a random walk on an appropriately *weighted* directed graph. Namely, if $w_{uv} \ge 0$ denote edge weights, a general probability transition matrix P can be defined as

$$P(u; v) = \frac{w_{uv}}{\sum_z w_{uz}}:$$

A probability distribution function $\pi : V(G) \to \mathbb{R}^+ \cup \{0\}$ satisfying $\sum_v \pi(v) = 1$ is said to be a *stationary distribution* of a random walk if

$$\pi P = \pi;$$

where π is viewed as a row vector.

In the case of undirected graphs, it can be easily shown that $\pi(v) = \frac{d(v)}{\sum_u d(u)}$ is a stationary distribution for a simple random walk on *any* undirected graph and is unique if the graph is connected. In general, there is no such closed formula for the stationary distribution in the directed case; nonetheless, a closed formula does exist for directed graphs in which the in-degree of each vertex is equal to its out-degree, called *Eulerian directed graphs*.

Example 1. Eulerian directed graphs have stationary distribution proportional to their out-degree sequences, $\pi(v) = \frac{d^+(v)}{\sum_u d^+(u)}$. Consequently, the stationary distribution of a directed regular graph with in-degrees and out-degrees all equal is given by the uniform distribution, $\pi = 1/n$.

Finding a closed formula for the stationary distribution of certain explicit families of directed graphs can be non-trivial, often requiring solving a set of recurrence relations. To illustrate this, we consider a "modified directed binary tree" example below and sketch steps for obtaining a closed formula for π.

Example 2. Let D be a modified perfect binary tree of height h with vertex set $V(D) = \{v_1, v_2, \ldots, v_{2^{h+1}-1}\}$ and edge set:

$$E(D) = \{(v_i, v_{2i}), (v_i, v_{2i+1}) : 1 \le i \le 2^h - 1\} \cup \{(v_i, v_{i+1}) : 2^h \le i \le 2^{h+1} - 2\} \cup \{(v_{2^{h+1}-1}, v_1)\}.$$

In other words, D is a perfect directed binary tree with a directed path across vertices in the bottom level leading back to the root. See Figure 1.1 for an illustration.

Ultimately, we can obtain a formula for π by analyzing the set of equations given by $xP = x$. We sketch the steps below:

Observe $x(2^{h+1} - 1) = x(1)$, $x(i) = x(j)$ for all i, j at the same depth, and $x(i) = \frac{x(j)}{2}$ if i has depth $k < h$ and j has depth $k - 1$.

For leaf vertices at depth h,

$$x(i+1) = \frac{x(2^h - 1)}{2} + x(i).$$

Figure 1.1 : The directed graph D in Example 2 for $h = 3$. The blue vertices have the maximum values in π while the red vertex has the minimum value in π.

Setting $x(2^h - 1) = 1$, solving the above recurrence, and letting $S = \sum_i x(i)$, we have

$$\pi(i) = \frac{1}{S}\begin{cases} 2^{h-j-1} & \text{if } i \text{ is at depth } j < h \\ \frac{i - 2^h + 1}{2} & \text{for } 2^h \le i \le 2^{h+1} - 1 \end{cases}.$$

In Example 2, the formula we obtained for π implies that the largest entry of π is 2^h times as large as the smallest; that is,

$$\frac{\max_i \pi(i)}{\min_i \pi(i)} = \frac{\pi(1)}{\pi(2^h)} = 2^h.$$

We remark that, in the undirected case, the closed formula for π ensures that all entries of the stationary distribution are within a factor of n, the number of vertices. However, in the directed case, this is not guaranteed. For instance, [15, Example 4] shows that the entries of the stationary distribution can be exponentially small in n for directed graphs. In general, Chung gives the bound:

Proposition 1 (Chung [15]). *For a strongly connected graph G on n vertices, the stationary distribution π of a random walk on G satisfies:*

$$\max_{i \in V(G)} \pi(i) \le k^D \min_{j \in V(G)} \pi(j);$$

where k is the maximum out-degree and D is the diameter of G.

Below, we sketch our own proof of this fact.

Proof. Since $(P^D)_{i;j} \geq \frac{1}{k^D}$, we have

$$\pi(i)\frac{1}{k^D} \leq \pi(i)(P^D)_{i;j} \leq \sum_{k \in V(G)} \pi(k)(P^D)_{k;j}$$
$$= (\pi P^D)_j = \pi(j).$$

Thus, for all $i; j \in V(G)$, we have:

$$\pi(i)\frac{1}{k^D} \leq \pi(j).$$

□

As we will explain further in Section 1.3.3, the extreme values of the stationary distribution play an important role in controlling the behavior of the random walk.

1.2.2 Perron-Frobenius theory and ergodicity

Two fundamental questions in the study of random walks concern the existence and uniqueness of a stationary distribution, as well as convergence to that distribution. Namely, a random walk is said to be *ergodic* if for any initial distribution f, the random walk converges to the unique stationary distribution π, i.e.,

$$\lim_{k \to \infty} fP^k = \pi.$$

For undirected graphs, the spectral decomposition of P shows a random walk is ergodic if and only if the graph is connected and non-bipartite (see [3] for further details). However, for directed graphs, no such closed formula exists for the stationary distribution and a more nuanced ergodicity criterion is required. The Perron-Frobenius theorem for non-negative matrices plays a central role in establishing conditions for ergodicity, as well as the existence of the stationary distribution.

Theorem 1 (Perron-Frobenius Theorem [25, 27]) Let A be non-negative matrix that is irreducible, i.e $(I + |A|)^{n-1} > 0$, with spectral radius $\rho(A)$. Then

(a) $(A) > 0$.

(b) (A) is an algebraically (and hence geometrically) simple eigenvalue of A.

(c) There are positive vectors x and y such that $Ax = (A)x$ and $y^T A = (A)y^T$.

From the definition of the probability transition matrix P, it is easy to see that $(P^k)_{u,v} > 0$ if and only if there exists a path of length k from u to v; hence strongly connected directed graphs have irreducible probability transition matrices. Furthermore, since $\sum_v P(u,v) = 1$ for each $u \in V$ of a strongly connected directed graph, $(P) \quad 1$ and

$$P\mathbf{1} = \mathbf{1};$$

and thus the all ones vector $\mathbf{1}$ is trivially the (right) Perron eigenvector associated with eigenvalue $1 = (P)$. By the Perron-Frobenius theorem, there exists a left (row) eigenvector with positive entries such that

$$P = :$$

We may scale so that $\sum_u (u) = 1$, in which case is the (unique) stationary distribution which we refer to as the Perron vector. However, as the following simple example shows, existence of the Perron vector for strongly connected directed graphs does not guarantee ergodicity:

Example 3. Labeling the vertices of a directed cycle $V = \{1, 2, 3\}$, let e_i be the probability distribution which places weight 1 on vertex i and 0 on all other vertices. Then

$$e_i P^k = e_{i+k};$$

where the indices are taken modulo 3.

Thus, we see that while a simple random walk on an undirected cycle of length 3 is ergodic, a simple random walk on a directed cycle of length 3 is not. In the above example, P^k oscillates between 3 transition matrices, making convergence to the stationary distribution impossible for other initial distributions. The period of this random walk is 3. In general, the period of a strongly connected directed graph is the number of eigenvalues of P with modulus 1; directed graphs with period 1 are *aperiodic*. In the language of directed graphs, the period is the greatest common divisor of the lengths of all its directed cycles. Having defined aperiodicity and irreducibility, we can now state the ergodicity criterion for random walks on directed graphs.

Theorem 2 (Ergodicity for random walks on directed graphs) A random walk on a directed graph G is ergodic if and only if G is strongly connected and aperiodic.

For completeness, we sketch an elementary proof of Theorem 2, which follows mainly from the following lemma:

Lemma 1. Let P be the probability transition matrix for a strongly connected, aperiodic directed graph G with associated Perron vector π. Then $\lim_{k\to\infty} P^k = 1\pi^T$.

Proof. To simplify notation, let $\Lambda = 1\pi^T$. Since $P1 = 1$, $\pi^T P = \pi^T$, and $1^T\pi = 1$, it follows immediately that, for $m = 1, 2, \ldots$

$$\Lambda^m = \Lambda; \tag{1.1}$$

$$P^m\Lambda = \Lambda P^m = \Lambda. \tag{1.2}$$

And, by induction, it easily follows from Eqs. (1.1) and (1.2) that

$$(P - \Lambda)^m = (P^m - \Lambda). \tag{1.3}$$

Now, note that if $(P - \Lambda)x = \lambda x$ for $\lambda \neq 0$, then by Eqs. (1.1) and (1.2), we have $\Lambda(P - \Lambda)x = 0$ so $\lambda\Lambda x = 0$, and thus $(P - \Lambda)x = Px = \lambda x$. Hence, if $(1, x)$ is an eigenpair for $(P - \Lambda)$ then $(1, x)$ is also an eigenpair for P. But, since P is an irreducible, non-negative matrix, the Perron-Frobenius theorem guarantees that $\rho(P) = 1$ has algebraic (and thus geometric) multiplicity of 1, so it must be that $x = c1$ for some scalar $c \neq 0$. This yields the contradiction

$$x = (P - \Lambda)x = (P - \Lambda)c1 = c1 - c1 = 0;$$

so 1 cannot be an eigenvalue of $(P - \Lambda)$. Ordering the eigenvalues of P by increasing modulus, $|\lambda_1| \leq \cdots \leq |\lambda_{n-1}| \leq |\lambda_n| = 1$, either $\rho(P - \Lambda) = 0$ or $\rho(P - \Lambda) = |\lambda_{n-1}|$. In either case,

$$\rho(P - \Lambda) \leq |\lambda_{n-1}| < 1.$$

Finally, since for any matrix $A \in M_n$, $\lim_{n\to\infty} A^n = 0$ if and only if $\rho(A) < 1$, combining the above with Eqn. (13) implies that as $m \to \infty$,

$$(P - \Pi)^m = (P^m - \Pi) \to 0.$$

□

Proof of Theorem 2. The sufficiency of the condition follows immediately from Lemma 1. If for any initial distribution f, $\lim_{k\to\infty} fP^k = \pi$, where $\pi > 0$ denotes the stationary distribution, then $P^k > 0$ for some k. Hence G must be strongly connected. Furthermore, by [27, Theorem 8.5.2], if A is a non-negative matrix with $A^k > 0$ for some k, then $\rho(A)$ is an algebraically simple eigenvalue; so G is aperiodic as well. □

Lastly, we note that the ergodicity criterion of irreducibility and aperiodicty can be characterized in matrix theoretic language by *primitivity*. A non-negative square matrix A is said to be *primitive* if there exists some positive natural number k such that $A^k > 0$.

Remark 1. A directed graph G is irreducible and aperiodic if and only if the probability transition matrix P of G is primitive.

1.3 Spectral graph theory

1.3.1 The normalized Laplacian

In addition to the probability transition matrix P, an important object of study in our analysis of random walks on directed graphs will be the directed normalized Laplacian matrix, as defined by Chung. Namely,

Definition 1 (Chung [15]). Let G be an n-vertex, strongly connected directed graph with associated probability transition matrix P. The normalized Laplacian L of G is

$$I - \frac{\Phi^{1/2} P \Phi^{-1/2} + \Phi^{-1/2} P^* \Phi^{1/2}}{2};$$

where $\Phi = \mathrm{diag}(\phi(1), \ldots, \phi(n))$ is a diagonal matrix with the entries of the Perron vector ϕ of P on the diagonal.

By construction, note that $\mathcal{L}^* = \mathcal{L}$ and hence $\mathcal{L}$ is Hermitian. If one takes G to be an undirected graph in the above definition, then applying the closed formula for the stationary distribution in the undirected case $\pi(u) = \frac{d(u)}{\sum_v d(v)}$ and examining $\mathcal{L}$ entry-wise, one can see the directed normalized Laplacian reduces to the undirected normalized Laplacian, defined by

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2};$$

where $D = \mathrm{diag}(d(1), \ldots, d(n))$ denotes the diagonal degree matrix and A denotes the adjacency matrix. We write the eigenvalues of $\mathcal{L}$ in increasing order, where

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}.$$

A useful tool in analyzing both the undirected and directed normalized Laplacian is the variational characterization of eigenvalues given by the Courant-Fischer theorem. In particular, the Courant-Fischer theorem characterizes the eigenvalues of a Hermitian matrix as solutions to optimization problems over subspaces S of fixed dimension.

Theorem 3 (Courant-Fischer [27]). For any Hermitian $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$,

$$\lambda_i = \min_{\substack{S \\ \dim(S) = i+1}} \left(\max_{\substack{x \in S \\ x \neq 0}} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} \right);$$

$$\lambda_i = \max_{\substack{S \\ \dim(S) = n-i}} \left(\min_{\substack{x \in S \\ x \neq 0}} \frac{\langle x, Ax \rangle}{\langle x, x \rangle} \right).$$

As a notable consequence of the Courant-Fischer theorem, the Rayleigh-Ritz theorem provides a simple expression for the largest and smallest eigenvalues.

Theorem 4 (Rayleigh-Ritz [27]). For any Hermitian $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$,

$$\lambda_0 = \min_{x \neq 0} \frac{\langle x, Ax \rangle}{\langle x, x \rangle};$$

$$\lambda_{n-1} = \max_{x \neq 0} \frac{\langle x, Ax \rangle}{\langle x, x \rangle}.$$

In the case of the normalized Laplacian, it is easy to see its eigenvalues are non-negative and $\lambda_0 = 0$ since $\mathcal{L}\Phi^{1/2}\mathbf{1} = 0$. We note that $R(A; x) := \frac{\langle x, Ax\rangle}{\langle x, x\rangle} = \frac{x^* Ax}{x^* x}$ is sometimes referred to as the *Rayleigh quotient*. Applying the Courant-Fischer theorem and rewriting the Rayleigh quotient for the directed normalized Laplacian $\mathcal{L}$, Chung [15] showed that

$$\lambda_1 = \inf_{\substack{g \in \mathbb{C}^n \\ \langle g, \Phi^{1/2}\mathbf{1}\rangle = 0}} \frac{\langle g, \mathcal{L} g\rangle}{\langle g, g\rangle} = \inf_{\substack{f \in \mathbb{C}^n \\ \sum_u f(u)\phi(u) = 0}} \frac{\sum_{u \to v} |f(u) - f(v)|^2 \phi(u) P(u, v)}{2 \sum_v |f(v)|^2 \phi(v)};$$

where $g = f\Phi^{1/2}$. In the undirected case, this characterization of λ_1 can be written as

$$\lambda_1 = \inf_{\substack{f \in \mathbb{C}^n \\ \sum_u f(u) d(u) = 0}} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v f(v)^2 d(v)}.$$

This first non-trivial eigenvalue, λ_1, is a key parameter for controlling a plethora of graph properties. In Section 1.3.3, we will see how λ_1 can be used to bound the rate of convergence of an ergodic random walk. Below, we describe the role of λ_1 in capturing isoperimetric properties of the graph.

1.3.2 Circulations and the Cheeger inequality

The classical isoperimetric problem in geometry concerns finding the maximum area-enclosing curve, among all curves of a given length. Isoperimetric problems in graphs can be framed analogously by measuring the "boundary" of a subset of vertices, taken to be the edges leaving that set, relative to some notion of the "size" of that set. In the case of undirected graphs, a notion of vertex subset size is given by *volume*, defined by

$$\mathrm{vol}(S) = \sum_{v \in S} d(v);$$

for some $S \subseteq V(G)$. However, there is no such natural notion of vertex degree or volume in the directed case, as vertices in a directed graph have both in-degree and out-degree, which may differ. Nonetheless, one can define a measure of volume in the directed case by using the stationary distribution of a random walk. This is achieved by considering a special type of flow on the edges of the directed graph called a circulation. More precisely, let $F : E(G) \to \mathbb{R}^+ \cup \{0\}$ denote a function assigning a non-negative value $F(u,v)$ to each edge (u,v) in a directed graph G. We call F a circulation if at each vertex v,

$$\sum_{u \in N^-(v)} F(u,v) = \sum_{w \in N^+(v)} F(v,w).$$

As shown in [15], one can associate a circulation F_ϕ with the left Perron vector ϕ of a probability transition matrix P by defining, for each $(u,v) \in E(G)$,

$$F_\phi(u,v) = \phi(u)P(u,v);$$

since

$$\begin{aligned}
\sum_{u \in N^-(v)} F_\phi(u,v) &= \sum_{u \in N^-(v)} \phi(u)P(u,v) \\
&= \phi(v) \\
&= \phi(v) \sum_{w \in N^+(v)} \frac{1}{d^+(v)} \\
&= \sum_{w \in N^+(v)} \phi(v)P(v,w) = \sum_{w \in N^+(v)} F_\phi(v,w).
\end{aligned}$$

Accordingly, the flow at a vertex v is given by its value in the stationary distribution, since

$$\phi(v) = \sum_{u \in N^-(v)} F_\phi(u,v) = \sum_{w \in N^+(v)} F_\phi(v,w).$$

We can now define the size of a vertex subset and its boundary using this notion of circulation. For a directed graph G, the out-boundary of $S \subseteq V(G)$, denoted ∂S, consists of all edges (u,v) with $u \in S$ and $v \notin S$. We define:

$$F(\partial S) = \sum_{u \in S, v \notin S} F(u,v);$$

$$F(S) = \sum_{v \in S} \sum_{u \in N(v)} F(u,v).$$

The Cheeger ratio $h(S)$ of a subset $S \subseteq V(G)$ is

$$h(S) = \frac{F(\partial S)}{\min\{F(S), F(\bar{S})\}}.$$

and the Cheeger constant of a directed graph G is $h(G) = \min_{S \subseteq V(G)} h(S)$. We note that the Cheeger constant is sometimes called *isoperimetric constant* or *conductance*. While computing the Cheeger constant for general families of graphs is not feasible in practice, the *Cheeger inequality* shows that normalized Laplacian eigenvalues can provide an estimate of $h(G)$. Namely, in the case of directed graphs, Chung proved:

Theorem 5 (Directed Cheeger inequality [15]) *If G is a directed graph with normalized Laplacian eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ and Cheeger constant $h(G)$, then*

$$\frac{h^2(G)}{2} \leq \lambda_1 \leq 2h(G).$$

We note that, in the undirected case, the Cheeger ratio of a subset is

$$h(S) = \frac{e(S,\bar{S})}{\min(\mathrm{vol}(S), \mathrm{vol}(\bar{S}))},$$

where $e(S,\bar{S})$ denotes the number of edges between S and its complement $\bar{S}$. Similarly, the (undirected) Cheeger constant is then the minimum of this Cheeger ratio over all vertex subsets and an identical statement for the Cheeger inequality in Theorem 5 holds in the undirected case. We note that the Cheeger inequality for the adjacency matrix of undirected graphs was first proved by Alon and Milman [4] and Tanner [49] for regular graphs. For general undirected graphs, see [14] for a proof of the undirected Cheeger inequality in terms of normalized Laplacian eigenvalues.

1.3.3 Bounding the rate of convergence

In Section 1.2.2, we reviewed the necessary and sufficient conditions for a random walk to converge to the unique stationary distribution. For such ergodic

Markov chains, a subsequent question, which has been the topic of much research (see [38] for a survey), is to determine how fast the random walks converges to the stationary distribution. In addressing this question, one can consider a variety of metrics to measure distance between the current and stationary distribution. For example, one might consider convergence in the standard Euclidean L_2 norm,

$$\Delta_{L_2}(s) = \max_f \| f P^s - \pi \| .$$

However, this metric may be considered weak for our purposes since it doesn't require convergence of the distribution at every vertex of the graph. A stronger, and perhaps more popular notion of convergence is given by total variation distance. Namely, the total variation distance Δ_{TV} after s steps is

$$\begin{aligned} \Delta_{TV}(s) &= \max_{A \subseteq V(G)} \max_{y \in V(G)} \sum_{x \in A} (P^s(y,x) - \pi(x)) \\ &= \frac{1}{2} \max_{y \in V(G)} \sum_{x \in V(G)} |P^s(y,x) - \pi(x)| . \end{aligned}$$

The χ-square distance is

$$\Delta'(s) = \max_{y \in V(G)} \left(\sum_{x \in V(G)} \frac{(P^s(y,x) - \pi(x))^2}{\pi(x)} \right)^{1/2} ,$$

and lastly, the relative pointwise distance simply determines the largest relative distance between the two distribution, i.e.

$$\Delta(s) = \max_{x,y} \frac{|P^s(x,y) - \pi(y)|}{\pi(y)} .$$

We note that convergence bounds for one of the above metrics may imply bounds for another; for example, since

$$\Delta_{TV}(s) \le \frac{1}{2} \Delta'(s) ,$$

we have that convergence upper bounds for $\Delta'(s)$ imply bounds for $\Delta_{TV}(s)$ as well. See [14] for further comparison of these metrics.

In deriving convergence bounds for the directed case, Chung considers a modified random walk called a *lazy random walk*. At each step in a lazy random walk, we stay at the current vertex with probability $1/2$, and with probability $1/2$, move to an out-neighbor of that vertex chosen uniformly at random. In other words, the transition probability matrix of a lazy random walk is

$$\tilde{P} = \frac{I + P}{2};$$

where P is the probability transition matrix of the simple random walk. Thus, one can think of a lazy random walk as a weighted random walk in which we add loops to each vertex. Consequently, a lazy random walk is always aperiodic, and hence, lazy random walks are ergodic for strongly connected directed graphs. In this way, lazy random walks allow us to relax the assumption of aperiodicity while preserving key spectral properties of P. Namely, note that the Perron vector ϕ of P is a left eigenvector of $\tilde{P}$ associated with eigenvalue 1. Furthermore, if P has left eigenvalues

$$\rho_0; \rho_1; \ldots; \rho_{n-1} = 1;$$

then $\tilde{P}$ has left eigenvalues $\frac{1+\rho_i}{2}$, where $\left|\frac{1+\rho_i}{2}\right| < 1$ since $\tilde{P}$ is aperiodic. Chung [15] proved the following theorem establishing an upper bound on the convergence rate for a lazy random walk on a directed graph.

Theorem 6 (Chung [15]). *Let G be a strongly connected directed graph on n vertices with normalized Laplacian eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ and Perron vector ϕ. Then after at most $s \geq 2\lambda_1^{-1}(-\log(\min_v \phi(v)) + 2c)$ steps of a lazy random walk on G, we have*

$$\Delta'(s) \leq e^{-c}.$$

Here, we see that normalized Laplacian eigenvalues and extreme values of ϕ play an important role in bounding the rate of convergence of random walks on directed graphs. This theorem serves as part of our motivation for studying the principal ratio, $\frac{\max_u \phi(u)}{\min_u \phi(u)}$, of the stationary distribution ϕ, as well as λ_1 of the normalized Laplacian $\mathcal{L}$, in Chapter 2.

1.4 Strong orientations of graphs

1.4.1 Preliminaries and Robbins' Theorem

The second half of this thesis concerns orientations of graphs, which naturally define a family of directed graphs called *oriented graphs*. Let G be a simple (undirected) graph with vertex set $V(G)$ and edge set $E(G)$. An orientation function of G is a sign-valued function σ on $\{(u,v),(v,u) : \{u,v\} \in E(G)\}$ that defines whether edge $\{u,v\} \in E(G)$ is oriented from u to v (in which case we write $u \to v$) or vice versa. More precisely,

$$\sigma(u,v) = \begin{cases} 1 & \text{if } u \to v \\ -1 & \text{if } v \to u \end{cases}.$$

The resulting directed graph D with vertex set $V(D) = V(G)$ and edge set $E(D) = \{(u,v) : \sigma(u,v) = 1\}$ is called an *orientation* of G. A directed graph is called an *oriented graph* if it is an orientation of a simple graph. Equivalently, oriented graphs are directed graphs without 2-cycles.

Our study of orientations of graphs will focus on the fundamental directed graph property of strong connectedness. We call a strongly connected orientation a *strong orientation*. A natural starting question in the study of strong orientations is characterizing existence of a strong orientation for a given undirected graph G. Here, the elegant Robbins' Theorem provides a simple criterion based on edge connectivity. Recalling that a connected graph $G = (V, E)$ is k-edge connected if G remains connected whenever fewer than k edges are removed from E, Robbins proved:

Theorem 7 (Robbins' Theorem [46]). *A graph G admits a strong orientation if and only if G is 2-edge connected.*

Alternatively stated, Robbins' Theorem states that graphs admitting strong orientations are precisely connected, *bridgeless* graphs, where a bridge is an edge whose removal increases the number of connected components of the graph. We remark that the necessity of the condition in Robbins' theorem is trivial, as it is clear that no orientation of a disconnected graph nor a graph featuring a bridge edge can yield a strongly connected directed graph. To prove the condition is sufficient, Robbins utilizes a tool called *ear decomposition*; we refer the reader to [46] for details. We note Robbins' theorem has since been extended to the broader setting of *mixed multigraphs* (see [6]), which are graphs whose edge sets E are in fact multisets, allowing for multiple edges between a pair of vertices, as well as both directed and undirected edges.

With regard to constructing strong orientations, linear-time algorithms which detect strong orientations and construct them whenever possible are known [20]. In particular, given an undirected graph G, a classic algorithm proceeds by depth-first search, orienting edges in the depth-first search tree from ancestor to descendant and, after all vertices have been uncovered, orienting any remaining edges from descendant to ancestor. See [47] for a proof that this orientation yields a strongly connected directed graph, provided the input graph is 2-edge connected. Just as Robbins' theorem has since been generalized to mixed multigraphs by Boesch and Tindell [6], Chung, Garey, and Tarjan [20] gave a linear-time algorithm that constructs strong orientations in mixed multigraphs.

1.4.2 Counting strong orientations

Although the existence and construction of strong orientations are well-understood topics, the task of *counting* strong orientations is less straightforward. The topic of counting strong orientations enjoys a multidisciplinary history. In fact, interest in counting strong orientations arose naturally in statistical mechanics in studying *ice-type models* used to study crystals with hydrogen bonds [33]. In these models, oxygen atoms form a square lattice, and the hydrogen ion between each pair of oxygen atoms is located in one of two positions: “close” or “far”. This configuration of hydrogen ions is said to satisfy Pauling's Ice Rule [43]. Roughly speaking, this states that of the four ions surrounding each atom, two are close and two are far, on their respective bond. In this way, one can naturally associate an Eulerian orientation (i.e. a strong orientation for which each vertex has equal in and out-degree) of a 4-regular graph with an allowable configuration of hydrogen ions. See Figure 1.2 for an example of this association.

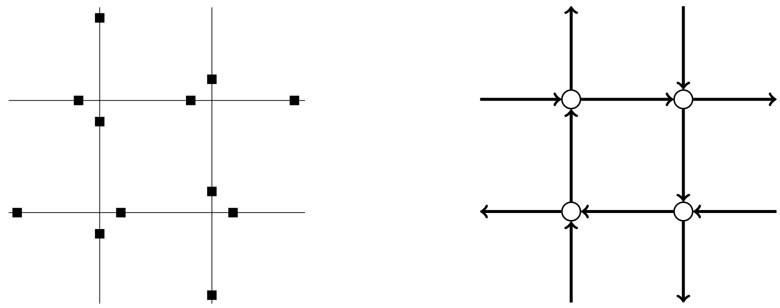

Figure 1.2 : Left: Hydrogen ion (squares) configuration satisfying the ice rule, where the intersection of two lines represents an oxygen atom. Right: the corresponding graph orientation, where vertices represent oxygen atoms, and edge (u, v) denotes that the hydrogen ion is close to v and far from u.

The total number of possible configurations, first determined by Lieb [33], is a key parameter in studying the *residual entropy* S of the model, defined as

$$S = k_B \ln(Z),$$

where k_B denotes Boltzmann's constant, Z denotes the allowable configurations.

More generally, the problem of counting the number of Eulerian and strong orientations of a given graph G is a special case of evaluating its Tutte polynomial, $T(G; x, y)$, defined recursively by

$$T(G; x, y) = \begin{cases} x \cdot T(G \backslash e; x, y) & \text{if } e \text{ is a bridge} \\ y \cdot T(G/e; x, y) & \text{if } e \text{ is a loop} \\ 1 & \text{if } E(G) = \emptyset \\ T(G \backslash e; x, y) + T(G/e; x, y) & \text{otherwise} \end{cases},$$

where for $e \in E(G)$, we let $G \backslash e$ denote the subgraph obtained from G by deleting the edge and G/e denote the subgraph obtained by contracting e (i.e., deleting $e = \{u, v\}$ and replacing $u, v \in V$ with a single vertex w). The Tutte polynomial captures a number of graph properties and also specializes to other well-known polynomials (e.g. along $xy = 1$, $T(G; x, y)$ specializes to the *Jones polynomial* of an alternating knot associated with graph G [50]). We refer the reader to [22, 53] for more discussion of the Tutte polynomial; here, we simply note that the number of Eulerian orientations of a given graph G is $T(G;0;-2)$ and number of strong orientations is $T(G;0;2)$ [51]. In general, counting Eulerian and strong orientations has been shown to be #P-hard (see [36] and [52] respectively), even for planar, bipartite graphs.

Instead of exact counting, other researchers have approximated the number of strong orientations for particular classes of graphs. In the case of α-dense graphs G (i.e. graphs with minimum degree $\delta(G) > \alpha n$ for $0 < \alpha < 1$), Alon, Frieze, and Welsh [39] developed a fully polynomial randomized approximation scheme for counting strong orientations. That is, they provided an algorithm that will, to an arbitrary degree of accuracy, approximate the number of strong orientations of G in polynomial time, depending on the size of G and the desired degree of accuracy. Nonetheless, the α-density assumption precludes sparse graphs from this scheme.

In Chapter 3, we will show how eigenvalues can reveal information about the number of strong orientations of a graph, even for possibly sparse, irregular graphs. Below, we give a complete overview of the remainder of this thesis.

1.5 Overview of main results

The remainder of this thesis is divided into three chapters, all of which concern problems in the spectral theory of directed graphs. In Chapter 2, we examine the stationary distribution of random walks on directed graphs, as well as λ_1 of the normalized Laplacian. In particular, we focus on the *principal ratio*, which is the ratio of maximum to minimum values of vertices in the stationary distribution. Here, our main results are:

- We give a sharp upper bound (Theorem 9, p. 24) for the principal ratio over all strongly connected graphs on n vertices. We explicitly compute the maximum principal ratio, characterize all graphs achieving this upper bound, and give explicit constructions for the extremal graphs (Theorem 10, p. 24).

- We show that under certain conditions, the principal ratio is tightly bounded (Theorem 11, p. 45). We also provide counterexamples (Examples 4–5, pp. 45 47) to show the principal ratio cannot be tightly bounded under weaker conditions.

We prove a lower bound (Theorem 15, p. 50) on $_1$, the rst nontrivial eigenvalue of the normalized Laplacian, for a strongly connected directed graph. We also give a construction (Example 6, p. 53) with $_1$ factorially small in the number of vertices.

In Chapter 3, we examine how eigenvalue conditions on an undirected graph may guarantee strong connectedness properties of orientations of that graph. Namely, we establish mild conditions under which a possibly irregular, sparse graph G has \many" strong orientations. Here, our main results are:

We show that under an isoperimetric condition and minimum degree requirement, a random orientation of G is strongly connected, with high probability (Theorem 16, p. 58). We show each condition is insu cient on its own in guaranteeing the result, and prove the minimum degree condition is tight (Proposition 16, p. 61), while the isoperimetric condition is almost tight (Proposition 17, p. 63).

We prove a related, but somewhat weaker version of the above theorem, replacing the isoperimetric condition with a condition on the spectral gap of the normalized Laplacian (Theorem 17, p. 75).

In Chapter 4, we conclude and explore a series of related, open problems for each result in this thesis. As partial progress towards Question 1, we compute the maximum hitting time between vertices in principal ratio extremal graphs (Claim 2, p. 85).

Chapter 2

Extreme values of the stationary distribution of random walks on directed graphs

2.1 Introduction

In the rst part of this chapter, we study extreme values of the stationary distribution of a random walk on a directed graph. In particular, we focus on the principal ratio (D) of a strongly connected directed graph D, de ned as

$$(D) = \frac{\max_u (u)}{\min_u (u)}:$$

As we saw in Section 1.3.3, the principal ratio has immediate implications for the central question of bounding the rate of convergence of a random walk on a directed graph, where extreme values of the stationary distribution play an important role in addition to eigenvalues (see Theorem 6). Another application of the stationary distribution and its principal ratio is in the algorithmic design and analysis of vertex ranking, particularly in so-called \PageRank' algorithms for directed graphs (since many real-world information networks are indeed directed graphs). PageRank algorithms [45] use a variation of random walks with an additional di usion parameter and therefore it is not surprising that the e ectiveness of the algorithm depends on the principal ratio. In addition to its role in Page Rank algorithmic analysis and bounding the rate of converge in random walks, it has been noted (see [21]) that the principal ratio can be interpreted as a numerical metric for graph irregularity since it achieves its minimum of 1 for regular graphs.

The study of the principal ratio of the stationary distribution has a rich history. We note that the stationary distribution is a special case of the Perron vector , which is the unique positive eigenvector associated with the largest eigenvalue of an irreducible matrix with non-negative entries. There is a large literature examining the Perron vector of the adjacency matrix of undirected graphs, which has been studied by Cioabă and Gregory [21], Tait and Tobin [48], Papendieck and Recht [42], Zhao and Hong [55], and Zhang [54].

For directed graphs, some relevant prior results are from matrix analysis. Latham [31], Minc [37], and Ostrowski [32] studied the Perron vector of a (not necessarily symmetric) matrix with positive entries, which can be used to study matrices associated with complete, weighted directed graphs. However, for our case, a more relevant prior result comes from Lynn and Timlake, who gave bounds of the principal ratio for primitive matrices with non-negative entries (see Corollary 2.1.1 in [35]). As we noted earlier, since ergodic random walks on directed graphs have primitive transition probability matrices, their result applies naturally in our setting.

Theorem 8 (Lynn, Timlake [35]). If A is an $n \times n$ nonnegative matrix satisfying $A^k > 0$ for some positive integer k and with Perron vector x (i.e. right eigenvector associated with the largest eigenvalue in modulus), then

$$\frac{\max_{1 \le i \le n} x_i}{\min_{1 \le i \le n} x_i} \le \frac{\lambda^k - m^{k-1}(r - m)}{m^k};$$

where $m = \min_{a_{ij} > 0} a_{ij}$, k is any integer such that $A^k > 0$, $r = \min_{1 \le i \le n} \sum_{j=1}^{n} a_{ij}$, and λ is the largest eigenvalue in absolute value.

This upper bound depends on the matrix A. To get an upper bound for the principal ratio of a strongly connected, aperiodic directed graph D, we can apply the above theorem with $A = P^T$, where P^T is the transpose of P, the probability transition matrix of a simple random walk on D. Namely, if we wish to get an absolute upper bound for all strongly connected, aperiodic directed graphs, we take = 1, m = $\frac{1}{n \quad 1}$, k n 1, and $\frac{1}{n \quad 1}$ r n. We get the following upper bound

$$\frac{\max_u \quad (u)}{\min_u \quad (u)} \quad K;$$

where

$$K \quad (n \quad 1)^{n \quad 1} \quad 1 \quad \frac{1}{(n \quad 1)^{n \quad 1}}(r \quad \frac{1}{n \quad 1})$$
$$= (1 + o(1))(n \quad 1)^{n \quad 1}:$$

Another prior bound on the principal ratio of directed graphs was given by Chung in [15]. In particular, the aforementioned Proposition 1 gives a bound on the principal ratio of a strongly connected directed graph that depends on certain graph parameters. Namely,

$$(D) \quad k^d;$$

where d is the diameter of the graph D and k is the maximum out-degree. Since d; k n 1, this bound also implies absolute upper bound on the principal ratio of $(n \quad 1)^{n \quad 1}$ over all strongly connected directed graphs on **n** vertices.

In this chapter, we provide an exact expression for the maximum of the principal ratio over all strongly connected directed graphs on **n** vertices. Asymptotically, our bound is

$$(n) = \max_{D : jV(D)j = n} \quad (D) = \quad \frac{2}{3} + o(1) \quad (n \quad 1)!:$$

Furthermore, we show that this bound is achieved by precisely three directed graphs, up to isomorphism.

In addition to an extremal analysis of the principal ratio, we also examine conditions under which the principal ratio can be tightly bounded. Namely, we show that if a directed graph satis es a degree condition and a discrepancy condition, then its principal ratio can be tightly bounded in the sense that it is \close" to the minimum possible value of 1. Furthermore, we provide counterexamples that show the principal ratio cannot be tightly bounded if either the discrepancy condition or degree conditions are removed.

2.2 A sharp upper bound on the principal ratio

We will prove an upper bound on the principal ratio in terms of n that is best possible. For $n \geq 3$, we define a function

$$\gamma(n) = \max\{\gamma(D) : D \text{ is a strongly connected } n\text{-vertex directed graph}\}.$$

Our main theorem is as follows.

Theorem 9. The maximum of the principal ratio of the stationary distribution over all strongly connected directed graphs on n vertices is asymptotically

$$\gamma(n) = \left(\frac{2}{3} + o(1)\right)(n-1)!.$$

This theorem is an immediate consequence of the following theorem which we prove.

Theorem 10. The maximum of the principal ratio of the stationary distribution over all strongly connected directed graphs on $n \geq 3$ vertices is exactly

$$\gamma(n) = \left(\frac{2}{3}\cdot\frac{n}{n-1} + \frac{1}{(n-1)!}\sum_{i=1}^{n-3} i!\right)(n-1)!.$$

Moreover, $\gamma(n)$ is attained only by directed graphs $D_1; D_2$; and D_3 defined as follows: $D_1; D_2$, and D_3 have vertex set $\{v_1; v_2; \dots; v_n\}$ and edge set

$$E(D) = \{(v_i; v_{i+1}) : 1 \leq i \leq n-1)\} \cup \{(v_j; v_i) : 1 \leq i < j \leq n-1\} \cup S(D);$$

where

$$S(D) = \begin{cases} \{(v_n; v_1)\} & \text{for } D = D_1; \\ \{(v_n; v_2)\} & \text{for } D = D_2; \\ \{(v_n; v_1); (v_n; v_2)\} & \text{for } D = D_3. \end{cases}$$

The case for $n = 5$ is illustrated in Figure 2:1.

We note that the extremal graphs $D_1; D_2; D_3$ are not only strongly connected, but also aperiodic. Thus, Theorem 1 still holds if one restricts attention to stationary distributions of ergodic random walks. We also remark that the graphs

Figure 2.1 : The three constructions D_1, D_2, D_3 for $n = 5$.

D_1 and D_2 are proper subgraphs of D_3. While all three graphs have different stationary distributions, their principal ratios are nonetheless equal.

The proof of Theorem 10 follows from a sequence of propositions. The basic idea is as follows: we first show that if the principal ratio of a directed graph achieves the bound in Theorem 10, then the graph must necessarily satisfy a set of properties, which are described in Section 2.2.1. In Sections 2.2.2 and 2.2.3, we identify families of graphs that satisfy these properties, but nonetheless are not extremal. Namely, given an arbitrary member from this family, we describe how one can modify this graph by adding or deleting edges so that its principal ratio strictly increases. In Section 2.2.4, we apply these propositions to show that unless a given graph is one of three graphs, it can be modified to increase its principal ratio. Finally, after establishing that all three of these extremal graphs indeed have the same principal ratio, we finish the proof and we explicitly compute the stationary distribution of one of these extremal graphs.

2.2.1 The structure of the extremal graphs

We assume all directed graphs D are strongly connected. For two vertices u and v, the distance $\mathrm{dist}(u, v)$ is the number of edges in a shortest directed path from u to v. For two subsets V_1, V_2, the directed distance $\mathrm{dist}(V_1, V_2)$ from V_1 to V_2 is defined as $\min\{\mathrm{dist}(u, v) : u \in V_1 \text{ and } v \in V_2\}$. For a directed graph D, let be the (left) eigenvector corresponding to the eigenvalue 1 for the transition probability matrix P. We de ne two subsets of $V(D)$ with respect to as follows.

$$V_{max} = \{v \in V(D) : \max_{u \in V(D)} (u) = (v)\}.$$

$$V_{min} = \{v \in V(D) : \min_{u \in V(D)} (u) = (v)\}.$$

We will establish a number of useful facts that relate the ratio of values of vertices of the Perron vector to the distance between those vertices.

Proposition 2. If $v_1, v_2, \ldots, v_k$ is a path of length k 1 from v_1 to v_k, then

$$\frac{(v_1)}{(v_k)} \quad \prod_{i=1}^{k\ 1} d^+(v_i).$$

Proof. From $P^k =$, we obtain

$$(v_k) = \sum_{z \in V(D)} (z)P^k(z, v_k) \qquad (v_1)P^k(v_1, v_k).$$

By considering the path $v_1, v_2, \ldots, v_k$, we have

$$P^k(v_1, v_k) \quad \prod_{i=1}^{k\ 1} \frac{1}{d^+(v_i)}.$$

Equivalently, $\frac{(v_1)}{(v_k)} \quad \prod_{i=1}^{k\ 1} d^+(v_i)$.

□

Proposition 3. If $\mathrm{dist}(u, v) = k$, then

$$\frac{(u)}{(v)} \quad (n \quad 1)_k,$$

where $(n \quad 1)_k = (n \quad 1) \ (n \quad 2) \quad (n \quad k)$ is the falling factorial.

Proof. Let $P = \{u = v_0, v_1, \ldots, v_k = v\}$ be a shortest path from u to v. For all $0 \quad i \quad k \quad 2$ and $j \quad i + 2$, we note that (v_i, v_j) is not a directed edge. Since D has no loops, we have $d^+(v_i) \quad n \quad k + i$ for all $0 \quad i \quad k \quad 1$. The proposition now follows by applying Proposition 2. □

Proposition 4. For any directed graph D with n vertices, we have $\mathrm{dist}(V_{max}, V_{min})$ $n \quad 2$.

Proof. Suppose dist$(V_{max}; V_{min}) = n \quad 1 = \text{dist}(u; v)$ for some u 2 V_{max} and v 2 V_{min}. Let $P = v_1; v_2; :::; v_n$ be a shortest directed path of length $n \quad 1$ such that $v_1 = u$ and $v_n = v$. Since P is a shortest directed path, we note v_2 is the only outneighbor of v_1. From $P = \quad$, we obtain

$$(v_2) = \quad (v_1) + \sum_{\substack{j \quad 3 \\ v_j \, ! \, v_2}} \frac{(v_j)}{d^+(v_j)}:$$

Thus $(v_2) \qquad (v_1)$ and so dist$(V_{max}; V_{min}) \qquad \text{dist}(v_2; v_n) \qquad n \quad 2$, which is a contradiction. □

Proposition 5. For a directed graph D with n vertices, if dist$(V_{max}; V_{min}) \quad n \quad 3$, then $(D) \quad \frac{1}{2}(n \quad 1)!$.

Proof. Let u 2 V_{max} and v 2 V_{min} such that dist$(u; v) = \text{dist}(V_{max}; V_{min})$. By Proposition 3, we have $(D) \quad (n \quad 1)_{n \quad 3} = \frac{1}{2}(n \quad 1)!$. □

Proposition 6. Let D be a strongly connected directed graph with vertex set $f v_1; :::; v_n g$. Assume $v_1; v_2; :::; v_n$ is a shortest directed path from v_1 to v_n. Suppose v_2 2 V_{max} and v_n 2 V_{min}. If $(D) > \frac{2}{3}(n \quad 1)!$, then we have $N^+(v_2) = f v_1; v_3 g$, $N^+(v_3) = f v_1; v_2; v_4 g$, and $d^+(v_i) \quad b \frac{2i}{3} c$ for $4 \quad i \quad n \quad 1$.

Proof. Since $v_1; :::; v_n$ is a shortest path from v_1 to v_n, we have $d^+(v_i) \quad i$. To prove $N^+(v_2) = f v_1; v_3 g$ and $N^+(v_3) = f v_1; v_2; v_4 g$, it therefore su ces to show $d^+(v_2) = 2$ and $d^+(v_3) = 3$. From $P = \quad$, we have for $1 \quad j \quad n \quad 1$,

$$(v_{j+1}) = \frac{(v_j)}{d^+(v_j)} + \sum_{\substack{i \quad j+2 \\ v_i \, ! \, v_j}} \frac{(v_i)}{d^+(v_i)} \qquad \frac{(v_j)}{d^+(v_j)} \qquad \frac{(v_j)}{j}:$$

If $d^+(v_2) = 1$, then applying the above bound we have $(v_n) \qquad \frac{(v_2)}{(n \quad 1):::4 \, 3}$, yielding the contradiction $(D) \quad \frac{1}{2}(n \quad 1)!$. Similarly, if $d^+(v_3) \quad 2$, or if $d^+(v_i) < b \frac{2i}{3} c$ for some i where $4 \quad i \quad n \quad 1$, then applying the above bound yields $(D) \quad \frac{2}{3}(n \quad 1)!$.

□

Proposition 7. Let D be a strongly connected directed graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Assume $v_2, \ldots, v_n$ is a shortest directed path from v_2 to v_n, where $v_2 \in V_{max}$ and $v_n \in V_{min}$ such that $dist(V_{max}, V_{min}) = n - 2$. If $\alpha(D) > \frac{2}{3}(n-1)!$, then we have $(v_1, v_2), (v_2, v_1) \in E(D)$ and v_2 is the only out-neighbor of v_1.

Proof. We first show (v_2, v_1) must be an edge. Suppose not. Then v_3 will be the only outneighbor of v_2. The equation $\pi P = \pi$ gives

$$\pi(v_3) = \pi(v_2) + \sum_{\substack{j \geq 4 \\ v_j \to v_3}} \frac{\pi(v_j)}{d^+(v_j)}.$$

Therefore, $\pi(v_3) \geq \pi(v_2)$ which yields that $v_3 \in V_{max}$ and $dist(V_{max}, V_{min}) \leq n-3$. By Proposition 5, we have $\alpha(D) \leq \frac{1}{2}(n-1)!$ which is a contradiction. Therefore, (v_2, v_1) is an edge.

Next, we will show $N^+(v_1) = \{v_2\}$. Since we assume $v_2, \ldots, v_n$ is a shortest path from v_2 to v_n, we have $N^+(v_2) = \{v_1, v_3\}$ and $N^+(v_1) \subseteq \{v_2, v_3, v_4\}$. Moreover, we have $d^+(v_i) \leq i$ for $3 \leq i \leq n$ as $N^+(v_i) \subseteq \{v_1, \ldots, v_{i-1}\} \cup \{v_{i+1}\}$. Lastly, we note that from $\pi P = \pi$, we have $\pi(v_1) \geq \frac{1}{2}\pi(v_2)$. Assume $v_4 \in N^+(v_1)$. Then by considering directed paths $v_1, v_4, \ldots, v_n$ and $v_2, v_3, \ldots, v_n$ and applying Proposition 2, we have

$$\begin{aligned} \pi(v_n) &\geq \frac{\pi(v_2)}{d^+(v_2) \cdots d^+(v_{n-1})} + \frac{\pi(v_1)}{d^+(v_1)\, d^+(v_4) \cdots d^+(v_{n-1})} \\ &\geq \frac{\pi(v_2)}{(n-1)!} + \frac{\pi(v_1)}{(n-1)_{n-3}} \geq \frac{2\pi(v_2)}{(n-1)!}, \end{aligned}$$

yielding the contradiction $\alpha(D) = \frac{\pi(v_2)}{\pi(v_n)} \leq \frac{1}{2}(n-1)!$. So, $N^+(v_1) \subseteq \{v_2, v_3\}$. Assume $v_3 \in N^+(v_1)$. Again, by considering directed paths $v_1, v_3, \ldots, v_n$ and $v_2, v_3, \ldots, v_n$ and applying Proposition 2, we similarly obtain

$$\begin{aligned} \pi(v_n) &\geq \frac{\pi(v_2)}{d^+(v_2) \cdots d^+(v_{n-1})} + \frac{\pi(v_1)}{d^+(v_1)\, d^+(v_3) \cdots d^+(v_{n-1})} \\ &\geq \frac{\pi(v_2)}{(n-1)!} + \frac{\pi(v_1)}{(n-1)_{n-2}} \geq \frac{3\pi(v_2)}{2(n-1)!}, \end{aligned}$$

yielding the contradiction $\gamma(D) = \frac{\phi(v_2)}{\phi(v_n)} \geq \frac{2}{3}(n-1)!$. Thus $v_3 \notin N^+(v_1)$ and since D is strongly connected, $N^+(v_1) \neq \emptyset$. Therefore, $N^+(v_1) = \{v_2\}$.

□

2.2.2 Adding edges to increase the principal ratio

Based on Propositions 1-6, we consider the definition of the following family of graphs. An extremal graph must satisfy (i)-(iv) in the definition below.

Definition 2. For each n, let $\mathcal{D}_n$ be a family of directed graphs where each $D \in \mathcal{D}_n$ on vertex set $\{v_1, \ldots, v_n\}$ satisfies the following properties:

(i) The shortest path from v_1 to v_n is of length $n-1$ and is denoted by $v_1, v_2, \ldots, v_n$.

(ii) For $i \in \{2, 3\}$, $d^+(v_i) = i$.

(iii) For each $4 \leq i \leq n-1$, we have $d^+(v_i) \geq \lfloor \frac{2i}{3} \rfloor$.

(iv) $v_2 \in V_{max}$, $v_n \in V_{min}$, and $\mathrm{dist}(V_{max}, V_{min}) = \mathrm{dist}(v_2, v_n) = n-2$.

(v) There exist i and j such that (v_j, v_i) is not an edge where $4 \leq j \leq n-1$ and $1 \leq i \leq j-1$.

For each $D \in \mathcal{D}_n$, we now define an associated graph D^+ identical to D except for the addition of a single edge.

Definition 3. For a given $D \in \mathcal{D}_n$, let $4 \leq t \leq n$ denote the smallest integer and $s < t$ the largest integer such that (v_t, v_s) is not an edge of D. Define D^+ as the directed graph with the same vertex set as D and with edge set $E(D) \cup \{(v_t, v_s)\}$, as illustrated in Figure 2.2.

For a given $D \in \mathcal{D}_n$, we wish to compare the principal ratios of D and D^+. In order to do so, we must establish some tools used to compare their stationary distributions. First, the following proposition provides a useful way to express entries of the Perron vector as a multiple of a single entry.

Figure 2.2 : D and D^+. A dashed edge indicates the absence of that edge.

Proposition 8. Let D be a directed graph whose vertex set is $\{v_1; :::; v_n\}$. We assume $v_1; :::; v_n$ is a shortest path from v_1 to v_n. If is the Perron vector of the transition probability matrix P, then for 1 i n, there exists a function f_i such that

$$(v_i) = f_i \quad (v_n);$$

where the the functions f_i satisfy

$$f_k = \frac{f_{k\;1}}{d^+(v_{k\;1})} + \sum_{\substack{i \quad k+1 \\ v_i \,!\, v_k}} \frac{f_i}{d^+(v_i)}: \qquad (2.1)$$

Proof. We proceed by induction. Trivially, $f_n = 1$. Let $1 < k < n \quad 1$. Assume the proposition holds for all integers j where $k \quad j \quad n$. We show the result holds for $i = k \quad 1$. As $= P$, we have

$$(v_k) = \frac{(v_{k\;1})}{d^+(v_{k\;1})} + \sum_{\substack{i \quad k+1 \\ v_i \,!\, v_k}} \frac{(v_i)}{d^+(v_i)}:$$

We note if $k = n$, then we do not have the second term of the equation above. Applying the induction hypothesis and rearranging the above yields

$$f_{k\;1} = d^+(v_{k\;1}) \left(f_k \quad \sum_{\substack{i \quad k+1 \\ v_i \,!\, v_k}} \frac{f_i}{d^+(v_i)} \right): \qquad (2.2)$$

□

The upshot of Proposition 8 is that when comparing two graphs D and D' where $V(D) = V(D') = \{v_1, \dots, v_n\}$ and $v_1, \dots, v_n$ is a shortest directed path from v_1 to v_n in D and D', we may write their Perron vectors entrywise as

$$\pi(v_i) = f_i \, \pi(v_n);$$
$$\sigma(v_i) = g_i \, \sigma(v_n);$$

for some functions f_i and g_i satisfying (2.1). The following proposition describes when $f_i = g_i$.

Proposition 9. Let D and D' and their respective Perron vectors be as described above.

If there is some $1 \le s \le n-1$ such that $d^+_D(v_i) = d^+_{D'}(v_i)$ for each $s \le i \le n$, then we have $f_i = g_i$ for each $s \le i \le n$.

This proposition can be proved inductively by using (2.1) and we skip the proof here. The next proposition compares f_i and g_i for the graphs D and D^+.

Proposition 10. For each $D \in \mathcal{D}_n$, let D^+ be as defined in Definition 3. Suppose π and σ are the Perron vectors of the transition probability matrices of D and D^+ respectively. Moreover, suppose $\pi(v_i) = f_i \, \pi(v_n)$ and $\sigma(v_i) = g_i \, \sigma(v_n)$ for each $1 \le i \le n$. We have

(a) $f_i = g_i$ for each $t+1 \le i \le n$.

(b) $\frac{g_t}{f_t} = \frac{d^+_D(v_t)+1}{d^+_D(v_t)}$.

(c) $\frac{g_{t-1} - f_{t-1}}{t-1} = \frac{g_t}{d^+_D(v_t)+1} = \frac{g_{t-2} - f_{t-2}}{(t-1)_2}$.

If $t \ge 5$, then additionally we have

(d) For each $3 \le k \le t-2$, we have $\frac{g_{t-k} - f_{t-k}}{(t-1)_k} \ge \frac{g_t}{d^+_D(v_t)+1}\left(1 - \frac{4}{3}\sum_{j=1}^{k-2} \frac{1}{(t-j)_2}\right) > 0$.

(e) For each $3 \le k \le t-2$, we have $\frac{g_{t-k} - f_{t-k}}{(t-1)_k} \le \frac{g_t}{d^+_D(v_t)+1}$.

Proof. Since $4 \le t \le n-1$ is the smallest integer such that an edge (v_t, v_s) is missing for some $1 \le s \le t-1$, we have $d^+(v_i) = i$ for each $2 \le i \le t-1$. We also note $d^+_D(v_i) = d^+_{D^+}(v_i)$ for each $1 \le i \ne t \le n$ and $d^+_D(v_t) + 1 = d^+_{D^+}(v_t)$.

Part (a) follows from Proposition 9 easily. Part (b) can be verified by using the equation (2.1). If $t \in \{3, 4\}$, then we do not need Part (d) or Part (e). We can compute Part (c) directly by using the out-degree conditions and the equation (2.1).

For Part (d) and Part (e), we first prove them simultaneously by induction on k for $3 \leq k \leq t - s - 1$. We mention here for the case where $k = t - s$, we will give the argument separately. If either $t = s + 1$ or $t = s + 2$, then we prove directly for $k = t - s$ and for $t - s + 1 \leq k \leq t - 2$ the proof is by induction.

The base case is $k = 3$. From (2.1), we have

$$g_{t-2} = \frac{g_{t-3}}{t-3} + \frac{g_{t-1}}{t-1} + \sum_{\substack{j \geq t \\ v_j \to v_{t-2}}} \frac{g_j}{d^+_{D^+}(v_j)};$$

$$f_{t-2} = \frac{f_{t-3}}{t-3} + \frac{f_{t-1}}{t-1} + \sum_{\substack{j \geq t \\ v_j \to v_{t-2}}} \frac{f_j}{d^+(v_j)}:$$

We note $\frac{f_j}{d^+_D(v_j)} = \frac{g_j}{d^+_{D^+}(v_j)}$ for all $j \geq t + 1$. Combining with Part (b), we have

$$g_{t-2} - f_{t-2} = \frac{g_{t-3} - f_{t-3}}{t-3} + \frac{g_{t-1} - f_{t-1}}{t-1}:$$

We solve for $\frac{g_{t-3} - f_{t-3}}{t-3}$ and divide both sides of the resulted equation by $(t-1)_2$. Then Part (c) gives the base case of Part (d) and Part (e).

For the inductive step, we assume Part (d) and Part (e) hold for all $3 \leq j \leq k - 1$. As for the base case, from equation (2.1), g_{t-k} satisfies the following equation:

$$g_{t-k} = \frac{g_{t-k-1}}{t-k-1} + \sum_{1 \leq j \leq k-1} \frac{g_{t-j}}{t-j} + \sum_{\substack{j \geq t \\ v_j \to v_{t-k}}} \frac{g_j}{d^+_{D^+}(v_j)}:$$

Similarly,

$$f_{t-k} = \frac{f_{t-k-1}}{t-k-1} + \sum_{1 \leq j \leq k-1} \frac{f_{t-j}}{t-j} + \sum_{\substack{j \geq t \\ v_j \to v_{t-k}}} \frac{f_j}{d^+_D(v_j)}:$$

Solving for $\frac{g_{t-k-1} - f_{t-k-1}}{t-k-1}$ and dividing both sides of the equation by $(t-1)_k$, we have

$$\frac{g_{t-k-1} - f_{t-k-1}}{(t-1)_{k+1}} = \frac{g_{t-k} - f_{t-k}}{(t-1)_k} - \sum_{j=1}^{k-1} \frac{g_{t-j} - f_{t-j}}{(t-j)(t-1)_k}:$$

We note $g_{t-j} - f_{t-j} > 0$ for each $1 \le j \le k-1$ by the inductive hypothesis of Part (d). Part (e) then follows from the inductive hypothesis of Part (e).

Applying Part (c) as well as the inductive hypothesis for Part (e), we have

$$\frac{g_{t-k-1} - f_{t-k-1}}{(t-1)_{k+1}} \geq \frac{g_t}{d_D^+(v_t)+1}\left(1 - \frac{4}{3}\sum_{j=1}^{k-2}\frac{1}{(t-j)_2} - \sum_{j=1}^{k-1}\frac{1}{(t-j)_{k-j+1}}\right);$$

since

$$\sum_{j=1}^{k-1}\frac{1}{(t-j)_{k-j+1}} = \frac{1}{(t-k+1)_2}\left(1 + \frac{1}{t-k+2} + \frac{1}{(t-k+3)_2} + \cdots + \frac{1}{(t-1)_{k-2}}\right)$$
$$\le \frac{1}{(t-k+1)_2}\sum_{j=0}^{\infty}\frac{1}{(t-k+2)^j}$$
$$< \frac{4}{3}\cdot\frac{1}{(t-k+1)_2};$$

we get

$$\frac{g_{t-k-1} - f_{t-k-1}}{(t-1)_{k+1}} \geq \frac{g_t}{d_D^+(v_t)+1}\left(1 - \frac{4}{3}\sum_{j=1}^{k-1}\frac{1}{(t-j)_2}\right).$$

We are left to show the expression in Part (d) is positive. We observe

$$\sum_{j=1}^{k-1}\frac{1}{(t-j)_2} \le \sum_{j=1}^{t-4}\frac{1}{(t-j)_2} = \frac{1}{(4)_2} + \cdots + \frac{1}{(t-1)_2} = \frac{1}{3} - \frac{1}{t-1} < \frac{1}{3};$$

here we used the assumption $n \ge 5$. We've completed the inductive step for Part (d).

An additional argument is needed for $k = t - s$ since $(v_t, v_s) \in E(D^+)$ and $(v_t, v_s) \notin E(D)$. We observe $s \geq 3$ since otherwise we do not need this argument. We have

$$g_s = \frac{g_{s-1}}{s-1} + \sum_{1 \le j \le s-t-1}\frac{g_{t-j}}{t-j} + \frac{g_t}{d_D^+(v_t)+1} + \sum_{\substack{j \ge t+1 \\ v_j \to v_s}}\frac{g_j}{d_D^+(v_j)};$$

while

$$f_s = \frac{f_{s-1}}{s-1} + \sum_{1 \le j \le s-t-1}\frac{f_{t-j}}{t-j} + \sum_{\substack{j \ge t+1 \\ v_j \to v_s}}\frac{f_j}{d_D^+(v_j)}.$$

As we did previously in the inductive proof, we have

$$\frac{g_{s-1}-f_{s-1}}{(t-1)^{t-s+1}} \geq \frac{g_t}{d^+(w_t)+1}\left(1-\frac{4}{3}\sum_{j=1}^{t-s-2}\frac{1}{(t-j)^2}-\sum_{j=1}^{t-s-1}\frac{1}{(t-j)^{t-s-j+1}}-\frac{1}{(t-1)^{t-s}}\right).$$

We need only to prove the first inequality of Part (d) for $k=t-s$. If $t-s=3$, then we prove Part (d) for $k=3$ directly. For $t-s\geq 4$, we have

$$\begin{aligned}\sum_{j=1}^{t-s-1}\frac{1}{(t-j)^{t-s-j+1}}+\frac{1}{(t-1)^{t-s}} &< \frac{1}{(s+1)^2}\left(\sum_{j=0}^{\infty}\frac{1}{(s+2)^j}+\frac{1}{(t-1)^{t-s-2}}\right)\\ &< \frac{1}{(s+1)^2}\left(\frac{5}{4}+\frac{1}{(s+2)(s+3)}\right)\\ &< \frac{4}{3}\cdot\frac{1}{(s+1)^2}.\end{aligned}$$

We used that $s\geq 3$ and $t-s\geq 4$ to prove the inequalities above. For the range of $t-s+1\leq k\leq t-2$, this can be proved along the same lines as the range of $3\leq k\leq t-s$. □

Using Proposition 10, we can now compare $\gamma(D)$ and $\gamma(D^+)$.

Proposition 11. For each $D\in\mathcal{D}_n$, let D^+ be defined as in Definition 3. Then $\gamma(D^+)>\gamma(D)$.

Proof. Since the Perron vector has positive entries, rescaling it by a positive number will not change the principal ratio. Thus we are able to assume ψ satisfies

$$\phi(v_2)=\psi(v_2).$$

To prove the claim, it is enough to show $\psi(v_n)>\phi(v_n)$. Suppose not, i.e., $\psi(v_n)\leq\phi(v_n)$.

Recall Proposition 10. If $t=3$ then we have $g_2>f_2$ as Part (a), Proposition 10. For $t=4$, we have $g_2>f_2$ as Part (b), Proposition 10. Since we assumed $\phi(v_n)\geq\psi(v_n)$, we have $\phi(v_2)=g_2\phi(v_n)>\psi(v_2)=f_2\psi(v_n)$, which is a contradiction. If $t\geq 5$, then we apply Part (d) of Proposition 10 with $k=t-2$ and get $g_2>f_2$. In the case of $t=5$, we still have the same inequality. Therefore, we can find the same contradiction as the case of $t=4$. □

Figure 2.3 : D and D^*. A dashed edge indicates the absence of that edge.

2.2.3 Deleting edges to increase the principal ratio

We now consider another family of graphs D'_n, disjoint from D_n, which satisfy the properties necessary for extremality in Section 2.2.1.

Definition 4. For each n, let D'_n be a family of directed graphs where each $D \in D'_n$ on vertex set $\{v_1, \ldots, v_n\}$ satisfies the following properties:

(i) The shortest path from v_1 to v_n is of length $n-1$ and is denoted by $v_1, v_2, \ldots, v_n$.

(ii) For each $2 \le i \le n-1$, $d^+(v_i) = i$.

(iii) $v_2 \in V_{max}$, $v_n \in V_{min}$, and $dist(V_{max}, V_{min}) = dist(v_2, v_n) = n-2$.

(iv) $d^+(v_n) \ge 2$.

(v) $N^+(v_n) \neq \{v_1, v_2\}$.

For each $D \in D'_n$, we now define an associated graph D^* identical to D except for the deletion of a single edge.

Definition 5. For each $D \in D'_n$, let $3 \le t \le n-1$ be the largest integer such that $(v_n, v_t) \in E(D)$. We define D^* as the directed graph whose edge set is $E(D) \setminus \{(v_n, v_t)\}$, as illustrated in Figure 2.3.

For a graph in D'_n, we consider the following algorithm.

Analogous to how Proposition 10 allowed us to compare the principal ratios of D and D^+, the following proposition will allow us to compare the principal ratios of D and D^-.

Proposition 12. For each $D \in D_n^0$, let D^- be defined as in Definition 5. Assume π and ψ are the Perron vectors of the transition probability matrices of D and D^- respectively. Moreover, suppose $\pi(v_i) = f_i\,\pi(v_n)$ and $\psi(v_i) = g_i\,\psi(v_n)$ for each $1 \le i \le n$. We have

(a) $f_i = g_i$ for $t \le i \le n$.

(b) $\frac{g_{t-1} - f_{t-1}}{t-1} = \frac{1}{d^+_D(v_n)}$.

(c) $0 < \frac{1}{d^+_D(v_n)}\left(1 - \frac{1}{(t-1)(d^+_D(v_n)-1)}\right) \le \frac{g_{t-2} - f_{t-2}}{(t-1)_2} \le \frac{1}{d^+_D(v_n)}$.

If $t \ge 5$, then additionally we have

(d) For $3 \le k \le t-2$, we have

$$\frac{g_{t-k} - f_{t-k}}{(t-1)_k} \ge \frac{1}{d^+_D(v_n)}\left(1 - \frac{4}{3}\sum_{j=1}^{k-2}\frac{1}{(t-j)_2}\right) - \frac{1}{d^+_D(v_n)-1}\sum_{j=1}^{k-1}\frac{1}{(t-1)_j} > 0;$$

(e) For $3 \le k \le t-2$, we have $\frac{g_{t-k} - f_{t-k}}{(t-1)_k} \le \frac{1}{d^+_D(v_n)}$:

Proof. We observed $d^+_D(v_i) = d^+_{D^-}(v_i) = i$ for each $1 \le i \le n-1$ and $d^+_D(v_n) - 1 = d^+_{D^-}(v_n)$. Also, $f_n = g_n = 1$. Part (a) is a simple consequence of Proposition 9. We can verify Part (b) and Part (c) directly. We note when we check Part (c), there are two cases depending on whether (v_n, v_{t-1}) is an edge or not. If $t \in \{3, 4\}$, then we do not need Part (d) or Part (e). Thus we assume $t \ge 5$. We will prove Part (d) and Part (e) simultaneously using induction.

The base case is $k = 3$. We have two cases.

<u>Case 1</u> $(v_n, v_{t-3}) \in E(D)$.

Using the equation (2.1), we have

$$g_{t-2} = \frac{g_{t-3}}{t-3} + \frac{g_{t-1}}{t-1} + \frac{1}{d^+_D(v_n) - 1} + \sum_{\substack{t \le j \le n-1 \\ v_j \to v_{t-3}}} \frac{g_j}{d^+_D(v_n)}: \qquad (2.3)$$

Similarly,

$$f_{t-2} = \frac{f_{t-3}}{t-3} + \frac{f_{t-1}}{t-1} + \frac{1}{d_D^+(v_n)} + \sum_{\substack{t \leq j \leq n-1 \\ v_j \to v_{t-3}}} \frac{f_j}{d_D^+(v_n)}. \qquad (2.4)$$

Subtracting f_{t-2} from g_{t-2}, rearranging terms followed by dividing both sides by $(t-1)_2$, we have

$$\frac{g_{t-3}-f_{t-3}}{(t-1)_3} = \frac{g_{t-2}-f_{t-2}}{(t-1)_2} - \frac{g_{t-1}-f_{t-1}}{(t-1)(t-1)_2} - \frac{1}{d_D^+(v_n)(d_D^+(v_n)-1)(t-1)_2}. \qquad (2.5)$$

Applying Part (a) to Part (c), we have

$$\frac{g_{t-3}-f_{t-3}}{(t-1)_3} \geq \frac{1}{d_D^+(v_n)}\left(1-\frac{1}{(t-1)_2}-\frac{1}{d_D^+(v_n)-1}\right)\left(\frac{1}{t-1}+\frac{1}{(t-1)_2}\right) > 0.$$

The above quantity is clearly positive since $t \geq 5$. Therefore, we obtained the base case for Part (d). From (2.5), if we apply Part (b) and Part (c) as well as $t \geq 5$, then we get $\frac{g_{t-3}-f_{t-3}}{(t-1)_3} \leq \frac{1}{d^+(v_n)}$, which is the base case for Part (e).

Case 2: $(v_n, v_{t-3}) \notin E(D)$.

If (v_n, v_{t-3}) is not an edge, then $\frac{1}{d_D^+(v_n)-1}$ is missing from (2.3) and $\frac{1}{d_D^+(v_n)}$ is missing in (2.4). However, (2.5) still holds in this case. We can prove the base case for Part (d) and Part (e) similarly.

For the inductive step, we assume Part (d) and Part (e) are true for all $3 \leq i \leq k$. We first deal with the case where (v_n, v_{t-k}) is an edge. Again, from equation (2.1) we have

$$g_{t-k} = \frac{g_{t-k-1}}{t-k-1} + \sum_{1 \leq j \leq k-1} \frac{g_{t-j}}{t-j} + \frac{1}{d_D^+(v_n)-1} + \sum_{t \leq j \leq n-1} \frac{g_j}{d_D^+(v_j)}. \qquad (2.6)$$

Similarly, for f_{t-k}, we have

$$f_{t-k} = \frac{f_{t-k-1}}{t-k-1} + \sum_{1 \leq j \leq k-1} \frac{f_{t-j}}{t-j} + \frac{1}{d_D^+(v_n)} + \sum_{t \leq j \leq n-1} \frac{f_j}{d_D^+(v_j)}. \qquad (2.7)$$

We solve for $\frac{g_{t-k-1}-f_{t-k-1}}{t-k-1}$ and then divide both sides of the equation by $(t-1)_k$. We get

$$\frac{g_{t-k-1}-f_{t-k-1}}{(t-1)_{k+1}} = \frac{g_{t-k}-f_{t-k}}{(t-1)_k} - \sum_{j=1}^{k-1} \frac{g_{t-j}-f_{t-j}}{(t-j)(t-1)_k} - \frac{1}{d_D^+(v_n)(d_D^+(v_n)-1)(t-1)_k}.$$

(2.8)

By the inductive hypothesis for Part (d) and Part (e), we get $\frac{g_{t-k-1}-f_{t-k-1}}{(t-1)^{k+1}} \geq \frac{g_{t-k}-f_{t-k}}{(t-1)^k} - \frac{1}{d^+(v_n)}$, which proves the inductive step for Part (e).

From the inductive hypothesis of Part (d), we get

$$\frac{g_{t-k}-f_{t-k}}{(t-1)^k} \geq \frac{1}{d_D^+(v_n)}\left(1-\frac{4}{3}\sum_{j=1}^{k-2}\frac{1}{(t-j)^2}-\frac{1}{d_D^+(v_n)-1}\sum_{j=1}^{k-1}\frac{1}{(t-1)^j}\right). \quad (2.9)$$

From the inductive hypothesis for Part (e), we have

$$\sum_{j=1}^{k-1}\frac{g_{t-j}-f_{t-j}}{(t-j)(t-1)^k} = \sum_{j=1}^{k-1}\frac{g_{t-j}-f_{t-j}}{(t-1)^j}\,\frac{1}{(t-j)^{k-j+1}} \leq \frac{1}{d^+(v_n)}\sum_{j=1}^{k-1}\frac{1}{(t-j)^{k-j+1}}. \quad (2.10)$$

Putting (2.8), (2.9) and (2.10) together, we get

$$\frac{g_{t-k-1}-f_{t-k-1}}{(t-1)^{k+1}} \geq \frac{1}{d_D^+(v_n)}\left(1-\frac{4}{3}\sum_{j=1}^{k-2}\frac{1}{(t-j)^2}-\sum_{j=1}^{k-1}\frac{1}{(t-j)^{k-j+1}} - \frac{1}{d_D^+(v_n)-1}\sum_{j=1}^{k}\frac{1}{(t-1)^j}\right).$$

By the same lines as the proof of Proposition 10, we can show $\sum_{j=1}^{k-1}\frac{1}{(t-j)^{k-j+1}} < \frac{4}{3}\frac{1}{(t-k+1)^2}$. Therefore, we proved

$$\frac{g_{t-k-1}-f_{t-k-1}}{(t-1)^{k+1}} \geq \frac{1}{d_D^+(v_n)}\left(1-\frac{4}{3}\sum_{j=1}^{k-1}\frac{1}{(t-j)^2}-\frac{1}{d_D^+(v_n)-1}\sum_{j=1}^{k}\frac{1}{(t-1)^j}\right).$$

We note

$$\frac{4}{3}\sum_{j=1}^{k-1}\frac{1}{(t-j)^2}+\frac{1}{d^+(v_n)-1}\sum_{j=1}^{k}\frac{1}{(t-1)^j} \leq \frac{4}{3}\cdot\frac{1}{3}\cdot\frac{1}{t-1}+\frac{1}{t-1}\sum_{i=0}^{\infty}\frac{1}{2^i} < \frac{4}{9}+\frac{3}{8} < 1.$$

Here we applied facts $t-k \geq 3$ and $t \geq 5$. Thus, that the expression in Part (d) is positive follows from the inequality above. We established the inductive step of Part (d) in the case where $(v_n; v_{t-k})$ is an edge. For the case where $(v_n; v_{t-k})$ is not an edge, we note $\frac{1}{d^+(v_n)-1}$ is missing from (2.6) and $\frac{1}{d^+(v_n)}$ is missing from (2.7). The argument goes along the same lines. □

Using Proposition 12, we can now compare (D) and $(D\ )$.

Proposition 13. For each $D \in D^0_n$, let D be defined as Definition 5. We have $(D\ ) > (D)$.

Proof. We use the same idea as the proof for Proposition 11. We rescale such that $(v_2) = (v_2)$ and show $(v_n) < (v_n)$. Suppose $(v_n)\ \ (v_n)$. We will show $g_2 > f_2$ which will yield $(v_2) > (v_2)$ since $(v_2) = g_2\ (v_n)$ and $(v_2) = f_2\ (v_n)$ as well as the assumption $(v_n)\ \ (v_n)$. If $t \in \{3; 4\}$, then $g_2 > f_2$ follows either from Part (b) or Part (c) of Proposition 12. If $t\ \ 5$, then we will apply Part (d) of Proposition 12 with $k = t\ \ 2$ to get $g_2 > f_2$. We draw the contradiction similarly. □

2.2.4 Proof of Theorem 10

We can now prove Theorem 10 as a consequence of Propositions 12, 13.

Proof of Theorem 10. We will show that the extremal graphs achieving the maximum of the principal ratio over all strongly connected n-vertex graphs are precisely $D_1; D_2$, and D_3 and that their principal ratio is indeed as claimed in Theorem 10.

We will use the fact that D_1 has principal ratio as follows, which we will prove at the end of this section:

$$(D_1) = \frac{2}{3}\left(\frac{n}{n\ \ 1} + \frac{1}{(n\ \ 1)!}\sum_{i=1}^{n\ \ 3} i!\right)(n\ \ 1)!:$$

Assume D is extremal, i.e. its principal ratio is at least as large as that of any directed graph on n vertices. For any (strongly connected) directed graph D, we have $\mathrm{dist}(V_{max}; V_{min})\ \ n\ \ 2$ by Proposition 4. If D is such that $\mathrm{dist}(V_{max}; V_{min})\ \ n\ \ 3$, then D is not extremal since by Proposition 5, we have $(D) < (D_1)$. So $\mathrm{dist}(V_{max}; V_{min}) = n\ \ 2$, where $v_2 \in V_{max}$, $v_n \in V_{min}$, and $v_2; v_3; \ldots; v_n$ is a shortest path from v_2 to v_n. If D is extremal, then $(D)\ \ (D_1) > \frac{2}{3}(n\ \ 1)!$. So, applying Proposition 7 and Proposition 6, we can assume further that $v_1; v_2; \ldots; v_n$ is a shortest path from v_1 to v_n, $d^+(v_i) = i$ for $i \in \{2; 3\}$, and $d^+(v_i)\ \ \lfloor\frac{2i}{3}\rfloor$ for $4\ \ i\ \ n$.

Now, if $D \in D_n$, then D is not extremal by Proposition 10. Similarly, if $D \in D^0_n$, D is not extremal by Proposition 12.

Therefore, $D \notin \mathcal{D}_n$ and $D \notin \mathcal{D}'_n$. Since $D \notin \mathcal{D}_n$ but satisfies all properties for inclusion in $\mathcal{D}_n$ except (v) in Definition 2, it must be that $d^+(v_i) = i$ for each $2 \le i \le n-1$. Then, since $D \notin \mathcal{D}'_n$ but satisfies all properties for inclusion in $\mathcal{D}'_n$ except either (iv) or (v) in Definition 4, either $d^+(v_n) = 1$ or $N^+(v_n) = \{v_1; v_2\}$. In the former case, if $N^+(v_n) = \{v_j\}$ for $j \ge 3$, then arguing along the same lines as in the proof of Proposition 13, one has $\rho(D) < \rho(D_1)$; otherwise $D = D_1$ or $D = D_2$. In the latter case, $D = D_3$.

Lastly, we show that $D_1; D_2$, and D_3 all have the same principal ratio. Assume $\alpha; \beta; \gamma$ are the Perron vectors of $D_1; D_2$, and D_3 respectively. Scale their Perron vectors so that all three agree on the nth coordinate. By Proposition 8, we know there exist (positive) functions $f_i; g_i; h_i$ so that

$$\alpha(v_i) = f_i \, \alpha(v_n);$$
$$\beta(v_i) = g_i \, \beta(v_n);$$
$$\gamma(v_i) = h_i \, \gamma(v_n):$$

By Proposition 9, we note $f_i = g_i = h_i$ for $2 \le i \le n$. We can prove the following inequalities for f_i.

(a) $\frac{f_{n-1}}{n-1} = \frac{f_{n-2}}{(n-1)_2} = f_n$.

(b) For each $3 \le k \le t-2$, we have $f_{n-1}\left(\frac{4}{3}\sum_{j=1}^{k-2}\frac{1}{(n-j)_2}\right) \le \frac{f_{n-k}}{(n-1)_k} \le f_n$:

The proof of Part (a) and Part (b) uses the same argument as the proof of Proposition 10 and it is omitted here. If $n \le 5$, then we can verify $\max\{f_i : 1 \le i \le n\} = f_2$ and $\min\{f_i : 1 \le i \le n\} = f_n$ directly. Suppose $n \ge 6$. By Part (b), for each $3 \le k \le n-2$ we have

$$\begin{aligned}
\frac{f_{n-k} - f_{n-k+1}}{(n-1)_k} &= \frac{f_{n-k}}{(n-1)_k} - \frac{f_{n-k+1}}{(n-1)_{k-1}(n-k)} \\
&\ge \frac{f_{n-k}}{(n-1)_k} - \frac{f_n}{n-k} \\
&\ge f_{n-1}\left(\frac{4}{3}\sum_{j=1}^{k-2}\frac{1}{(n-j)_2} - \frac{1}{n-k}\right) \\
&\ge f_{n-1}\left(\frac{4}{3(n-k+2)} - \frac{1}{n-k}\right)
\end{aligned}$$

$$> f_n \left(1 - \frac{1}{3} - \frac{1}{2}\right) = \frac{f_n}{6}.$$

We note $n \geq k \geq 2$. We can check $f_1 > f_n$ easily. Therefore, we obtain $\max\{f_i : 1 \leq i \leq n\} = f_2$ and $\min\{f_i : 1 \leq i \leq n\} = f_n$.

The same holds for g_2 and h_2, which completes the proof. □

We now compute the stationary distribution and principal ratio of D_1, completing the proof of Theorem 10.

Claim A. Let D_1 be as defined in the statement of Theorem 10, and let π be the Perron vector associated with the transition probability matrix P of D_1. Then

$$\gamma(D_1) = \frac{2}{3}\left(\frac{n}{n-1} + \frac{1}{(n-1)!}\left(\sum_{i=1}^{n-3} i! - (n-1)!\right)\right),$$

where

$$\min_{1 \leq i \leq n} \pi(v_i) = \pi(v_n) \text{ and } \max_{1 \leq i \leq n} \pi(v_i) = \pi(v_2).$$

Proof. Since we are concerned with the ratio of the maximum entry and the minimum entry of the Perron vector, rescaling the Perron vector by a positive number will not affect our result. We assume $x = (x_1, x_2, \ldots, x_n)$ with $x_n = 1$ such that $xP = x$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1/2 & 0 & 1/2 & \cdots & 0 & 0 \\ 1/3 & 1/3 & 0 & \cdots & 0 & 0 \\ 1/4 & 1/4 & 1/4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & 0 & \frac{1}{n-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Suppose $P = (p_1, p_2, \ldots, p_n)$ where p_i is the i-th column of P for each $1 \leq i \leq n$. From $x_1 = x \cdot p_1$ and $x_2 = x \cdot p_2$, we have

$$x_2 = \frac{4}{3}x_1 - \frac{2}{3}, \tag{2.11}$$

where we used the assumption $x_n = 1$. As $x_3 = x - p_3$ and $x_1 = x - p_1$, we have

$$x_3 = \frac{3}{4}x_1 - \frac{3}{4}. \tag{2.12}$$

For each $2 \leq k \leq n-1$, we define

$$a_k = \frac{2k}{(k+1)(k-1)!},$$

$$b_k = \frac{k}{(k+1)(k-1)!}\sum_{i=0}^{k-2} i!.$$

For each $2 \leq k \leq n-1$, we will show

$$x_k = a_k x_1 - b_k. \tag{2.13}$$

We will prove (2.13) by induction on k. The cases $k = 2$ and $k = 3$ are given by (2.11) and (2.12) respectively. Assume (2.13) is true up to l for some $3 \leq l \leq n-2$. Using $x_{l+1} = x - p_{l+1}$ and $x_{l-1} = x - p_{l-1}$, we have

$$\begin{aligned}
x_{l+1} &= \frac{l+1}{l+2}\left(x_{l-1} - \frac{x_{l-2}}{l-2}\right) \\
&= \frac{l+1}{l+2}\left((a_{l-1}x_1 - b_{l-1}) - \frac{a_{l-2}x_1 - b_{l-2}}{l-2}\right) \\
&= \frac{l+1}{l+2}\left(a_{l-1} - \frac{a_{l-2}}{l-2}\right)x_1 - \frac{l+1}{l+2}\left(b_{l-1} - \frac{b_{l-2}}{l-2}\right) \\
&= a_{l+1}x_1 - b_{l+1}.
\end{aligned}$$

The inductive hypothesis and an elementary computation gives:

$$\begin{aligned}
a_{l+1} &= \frac{l+1}{l+2}\left(a_{l-1} - \frac{a_{l-2}}{l-2}\right) \\
&= \frac{l+1}{l+2}\left(\frac{2(l-1)}{l(l-2)!} - \frac{2(l-2)}{(l-1)!}\right) \\
&= \frac{2(l+1)}{(l+2)\,l!}.
\end{aligned}$$

$$b_{l+1} = \frac{l+1}{l+2}\left(b_{l-1} - \frac{b_{l-2}}{l-2}\right)$$

$$= \frac{l+1}{(l+2)}\left(\frac{(l-1)^2}{l!}\sum_{i=0}^{l-3} i! - \frac{l(l-2)}{l!}\sum_{i=0}^{l-4} i!\right)$$
$$= \frac{l+1}{(l+2)\,l!}\left(l(l-2)\left(\sum_{i=0}^{l-3} i! - \sum_{i=0}^{l-4} i!\right) + \sum_{i=0}^{l-3} i!\right)$$
$$= \frac{l+1}{(l+2)\,l!}\left(l(l-2)! + \sum_{i=0}^{l-3} i!\right)$$
$$= \frac{l+1}{(l+2)\,l!}\sum_{i=0}^{l-1} i!.$$

We have completed the proof of (2.13). Since $x_n = x - p_n$, we have

$$x_n = \frac{x_{n-1}}{n-1}. \qquad (2.14)$$

Recall the assumption $x_n = 1$. Using (2.13) with $k = n-1$ and solving for x_1 in (2.14), we obtain

$$x_1 = \frac{n(n-2)!}{2} + \frac{1}{2}\sum_{i=0}^{n-3} i!.$$

We already have an explicit expression for entries of x. We claim

$$x_2 > x_1 > x_3 > x_4 > \cdots > x_{n-1} > x_n.$$

We can verify $x_2 > x_1 > x_3$ and $x_{n-1} > x_n$ directly. To prove the remaining inequalities, for each $3 \le k \le n-2$, (2.13) yields

$$x_k = \frac{2k}{(k+1)(k-1)!}x_1 - \frac{k}{(k+1)(k-1)!}\sum_{i=0}^{k-2} i!; \qquad (2.15)$$
$$x_{k+1} = \frac{2(k+1)}{(k+2)\,k!}x_1 - \frac{(k+1)}{(k+2)\,k!}\sum_{i=0}^{k-1} i!. \qquad (2.16)$$

We first multiply (2.16) by a factor $-\frac{k^2(k+2)}{(k+1)^2}$ and add the resulting equation to (2.15). We get the following equation

$$x_k - \frac{k^2(k+2)}{(k+1)^2}x_{k+1} = \frac{k}{k+1}.$$

The equation above implies $x_k > x_{k+1}$ for each $3 \le k \le n-2$. We have finished the proof of the claim. As the Perron vector ν is a positive multiplier of x, we have

$$\min_{1\le i\le n} \nu(v_i) = \nu(v_n) \text{ and } \max_{1\le i\le n} \nu(v_i) = \nu(v_2).$$

Finally, we are able to compute

$$
\begin{aligned}
\frac{\phi(v_2)}{\phi(v_n)} &= \frac{x_2}{x_n} \\
&= \frac{2n(n-2)!}{3} + \frac{2}{3}\sum_{i=0}^{n-3} i! - \frac{2}{3} \\
&= \frac{2}{3}\left(\frac{n}{n-1} + \frac{1}{(n-1)!}\sum_{i=1}^{n-3} i!\right)(n-1)!.
\end{aligned}
$$

This completes the proof of Theorem 10. □

2.3 A sufficient condition for a tightly bounded principal ratio

So far, we have shown that the maximum of the principal ratio over all strongly connected n-vertex directed graphs is $(2/3 + o(1))(n-1)!$. On the other hand, the minimum of the principal ratio is 1 and is achieved by regular directed graphs. In this section, we examine conditions under which the principal ratio is "close" to the minimum of 1.

An important tool in our analysis will be the aforementioned notion of circulation, as defined by Chung [15]. Recall that for a directed graph D, a function $F : E(D) \to \mathbb{R}^+ \cup \{0\}$ that assigns to each directed edge (u,v) a nonnegative value $F(u,v)$. F is said to be a *circulation* if at each vertex v, we have

$$
\sum_{u: u \in N^-(v)} F(u,v) = \sum_{w: w \in N^+(v)} F(v,w).
$$

For a circulation F and a directed edge $e = (u,v)$, we will write $F(e)$ for $F(u,v)$ in some occasions. If ϕ is the Perron vector of the transition probability matrix P, then we can associate a circulation F_ϕ to ϕ, where

$$
F_\phi(v,w) = \frac{\phi(v)}{d^+(v)}.
$$

In particular, we recall that the circulation F_ϕ has the following property: at each vertex v, we have

$$
\sum_{u: u \in N^-(v)} F_\phi(u,v) = \phi(v) = \sum_{w: w \in N^+(v)} F_\phi(v,w). \tag{2.17}
$$

We will repeatedly use (2.17) in the proof of the following theorem.

Theorem 11. Let $D = (V, E)$ be a strongly connected directed graph and ϕ be the Perron vector of the transition probability matrix P. If there are positive constants a, b, c, d, ϵ such that

(i) $(a-\epsilon)n \le d^+(v), d^-(v) \le (a+\epsilon)n$ for all $v \in V(D)$ and

(ii) $|E(S,T)| \ge b|S||T|$ for all disjoint subsets S and T with $|S| \ge cn$ and $|T| \ge dn$,

then we have

$$\gamma(D) \le \frac{1}{C} \text{ for } C = \frac{b(a-5\epsilon)(a-\epsilon)}{4(a+\epsilon)^2}.$$

2.3.1 Discussion of conditions

Before proceeding with the proof of Theorem 11, we illustrate that neither the degree condition (i), nor the discrepancy condition (ii) alone guarantee a small principal ratio. We first give a construction which satisfies the degree requirement but fails the discrepancy condition and has principal ratio linear in n.

Example 4. Construct a directed graph D on $2n+1$ vertices as follows: take two copies of $\vec{D}_n$, the complete directed graph on n vertices, as well as an isolated vertex b. Add an edge from each vertex in the first copy of $\vec{D}_n$ to b and an edge from b to each vertex in the second copy of $\vec{D}_n$. Finally, select a distinguished vertex from the first copy of $\vec{D}_n$, which we denote e, and a distinguished vertex from the second copy of $\vec{D}_n$, which we denote d, and add edge (d, e). Let A denote the induced subgraph of the first copy of $\vec{D}_n$ obtained by deleting vertex e; similarly, C is the induced subgraph obtained by deleting vertex d from the second copy of $\vec{D}_n$. See Figure 2.4 for an illustration.

Proposition 14. The construction D in Example 4 satisfies the degree condition of Theorem 11 but not the discrepancy condition. The (unscaled) Perron vector of

Figure 2.4 : The construction in Example 4.

D is given by

$$(u) = \begin{cases} 1 & u \, 2 \, V(A) \\ \frac{n+1}{n} & u = b \\ \frac{(n+1)^2(n \;\; 1)}{n^2} & u \, 2 \, V(C) \\ n+1 & u = d \\ 2 & u = e \end{cases} :$$

Consequently, $(D) = \frac{\max_u \;\; (u)}{\min_u \;\; (u)} = n + 1$.

Proof. Observe that, for all $a \, 2 \, V(A)$, $d^+_a = d^+_b = d^+_d = d^+_e = n$, and, for all $c \, 2 \, V(C)$, $d^+_c = n \;\; 1$, thus D satis es the degree condition in Theorem 11. However, D fails the discrepancy condition since $E(V(A); V(C)) = 0$ where $jV(A)j = jV(C)j = n \;\; 1$. To compute the Perron vector of D, rst observe that since A and C are vertex-transitive, $(u) = \;\; (a)$ for all $u; a \, 2 \, V(A)$ and similarly $(u) = \;\; (c)$ for all $u; c \, 2 \, V(C)$. Consider $a \, 2 \, V(A)$. From $\;\; = P$, we obtain

$$\begin{aligned} (a) &= \sum_{u 2 N \;\; (a)} (u) P(u; a) \\ &= \sum_{u 2 N \;\; (a) n V(A)} (u) P(u; a) + \sum_{u 2 V(A)} (u) P(u; a) \\ &= \frac{(e)}{d^+_e} + \sum_{u 2 V(A)} \frac{(a)}{d^+_a} \\ &= \frac{(e)}{n} + \frac{n \;\; 2}{n} \; (a): \end{aligned}$$

In the same way as above, we also obtain equations for vertices $b; d; e$ and $c \, 2 \, C$:

$$(b) = \frac{n \;\; 1}{n} \; (a) + \frac{(e)}{n};$$

$$(c) = \frac{(b)}{n} + \frac{n-2}{n-1}(c) + \frac{(d)}{n};$$
$$(d) = \frac{(b)}{n} + (c);$$
$$(e) = \frac{n-1}{n}(a) + \frac{(d)}{n}:$$

We may set $(a) = 1$ and solve the above equations, yielding the result. □

Next, we give a construction to illustrate the discrepancy condition alone is insufficient to guarantee a small principal ratio.

Example 5. Construct a directed graph D on $n + \sqrt{n}$ vertices as follows: first, construct the following graph from [15] on $\sqrt{n}$ vertices, which we denote $H_{\sqrt{n}}$. To construct $H_{\sqrt{n}}$, take the union of a directed cycle $C_{\sqrt{n}}$ consisting of edges $(v_j; v_{j+1})$ (where indices are taken modulo $\sqrt{n}$), and edges $(v_j; v_1)$ for $j = 1; \ldots; \sqrt{n} - 1$. Then, take a copy of D_n, the complete directed graph on vertices, and select from it a distinguished vertex u. Add edges $(v_1; u)$ and $(u; v_1)$. See Figure 2.5 for an illustration.

Figure 2.5 : The construction in Example 5.

It is easy to check D as defined in Example 5 satisfies the discrepancy condition in Theorem 11, but not the degree requirement (note $d^+_{v_{\sqrt{n}}} = 1$ and $d^+_u = n$). As noted in [15], the graph $H_{\sqrt{n}}$ has principal ratio $2^{\sqrt{n}-1}$. Thus, $(D) \quad (H_{\sqrt{n}}) = 2^{\sqrt{n}-1}$.

2.3.2 Proof of Theorem 11

Having shown that each condition in Theorem 11 taken on its own is insufficient in ensuring a small principal ratio, we now prove that together they do provide a sufficient condition.

Proof of Theorem 11. We assume

$$\max_{v\in D(V)} (v) = (u) \text{ and } \min_{v\in D(V)} (v) = (w):$$

We will show (w) C (u) instead, where C is the constant in the statement of the theorem. We use U to denote the set $\{v \in N(u) : (v) \frac{(u)}{2}\}$. If $w \in N(u)\setminus U$, then we have nothing to show. Thus we assume $w \notin N(u)\setminus U$. We consider the circulation F associated with and recall (2.17). By the definition of U, we have

$$(u) = \sum_{v\in N(u)} F(v;u) = \sum_{v\in U} F(v;u) + \sum_{v\in N(u)\setminus U} F(v;u)$$
$$\sum_{v\in U} \frac{(u)}{2(a\ )n} + \sum_{v\in N(u)\setminus U} \frac{(u)}{(a\ )n}$$
$$\frac{|U|\ (u)}{2(a\ )n} + \frac{((a+\ )n\ |U|)\ (u)}{(a\ )n}:$$

Solving the inequality above, we have $|U|\ 4n$. Let $U' = N(u)\setminus U$. Then we have $|U'|\ (a\ 5\ )n$ from the assumption $|N(u)|\ (a\ )n$. If $|N(w)\cap U'|\ \frac{|U'|}{2}$, then we have

$$(w) = \sum_{v\in N(w)} F(v;w)$$
$$\sum_{v\in N(w)\cap U'} F(v;w)$$
$$\sum_{v\in N(w)\cap U'} \frac{(u)}{2(a+\ )n}$$
$$\frac{(a\ 5\ )\ (u)}{4(a+\ )}$$
$$C\ (u):$$

Therefore, we assume $|N(w)\cap U'| < \frac{|U'|}{2}$ for the remainder of the proof. We define $U'' = U'\setminus N(w)$ and we have $|U''|\ \frac{(a\ 5\ )n}{2}$. The assumption $|E(S;T)|\ b|S||T|$ for any disjoint S and T implies

$$|E(U'';N(w))|\ b|U''||N(w)|\ \frac{b(a\ 5\ )(a\ )n^2}{2}: \qquad (2.18)$$

Set $_1 = \sum_{v \in N(w)} (v)$ and $E_1 = E(U^{00}; N(w))$. Using (2.17), we have the following inequality

$$_1 = \sum_{v \in N(w)} \sum_{z \in N(v)} F(z;v) \quad \sum_{e \in E_1} F(e) \quad \sum_{e \in E_1} \frac{(u)}{2(a+\ )n} \quad C(a+\ )n\ (u); \tag{2.19}$$

where we used inequality (2.18) in the last step. By the de nition of the circulation F , we have

$$(w) = \sum_{v \in N(w)} F(v;w) \quad \sum_{v \in N(w)} \frac{(v)}{(a+\ )n} = \frac{1}{(a+\ )n}: \tag{2.20}$$

The combination of inequalities (2.19) and (2.20) now completes the proof. □

2.4 Bounds on the rst non-trivial eigenvalue, $_1$

In this section, we focus on the rst non-trivial eigenvalue of the normalized Laplacian. As we saw in Theorems 6 and 5, this eigenvalue $_1$ is a key parameter in bounding the rate of convergence of random walks on directed graphs and in capturing isoperimetric properties of the directed graph. In the undirected case, it is well-known that $_1$ and the diameter of the graph G are intimately related. Speci cally, one can derive lower bounds on $_1$ in terms of the diameter of the graph and, conversely, derive upper bounds on the diameter in terms of $_1$. As an example of the former, consider the following bound from [14]:

Theorem 12 ([14]). For a connected graph G with diameter D and normalized Laplacian eigenvalues $0 = \ _0 \quad _1 \quad \quad _{n\ 1}$, we have

$$_1 \quad \frac{1}{D\ \mathrm{vol}(G)}:$$

For more specialized classes of graphs (e.g. for vertex-transitive and edge-transitive graphs), one can derive tighter bounds by applying lower bounds on the Cheeger constant with the Cheeger inequality (see [14, Theorems 7.5{7.7]); however, the above bound is asymptotically sharp up to a constant in general. Additionally, Chung proved the following upper bound on graph diameter in terms of $_1$.

Theorem 13 (Chung [14, 19]) For a connected graph G on n vertices with diameter D and normalized Laplacian eigenvalues $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$, we have

$$D \le \left\lceil \frac{\log(n-1)}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \right\rceil + 1.$$

In the directed case, Chung established a result similar to Theorem 13.

Theorem 14 (Chung [15]). For a strongly connected directed graph G on n vertices with diameter D, normalized Laplacian eigenvalues $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$, and Perron vector ϕ of the probability transition matrix of a random walk on G, we have

$$D \le \left\lceil \frac{2 \max_u \log(1/\phi(u))}{\log \frac{2}{2-\lambda_1}} \right\rceil + 1.$$

In this section, we prove a lower bound for λ_1 for a strongly connected directed graph, which can be thought of as the directed analog to Theorem 12. We also investigate the sharpness of this bound by constructing an example with small λ_1. Our lower bound is as follows:

Theorem 15. For a strongly connected directed graph G on n vertices with diameter D, normalized Laplacian eigenvalues $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$, and Perron vector ϕ of the probability transition matrix of a random walk on G, we have

$$\lambda_1 > \frac{\min_u \phi(u)}{2D \cdot \max_u d^+(u)}.$$

2.4.1 Proof of Theorem 15

Before proceeding with the proof of Theorem 15, we establish the following useful fact.

Fact 1. Let $f \in \mathbb{C}^n$ and $g \in (\mathbb{R}^+)^n$ such that $\sum_u f(u)g(u) = 0$. Then, for every u there exists v such that $|f(u) - f(v)| > |f(u)|$.

Proof. For each $j = 1, \ldots, n$, let $x(j) = [\mathrm{Re}(f(j)), \mathrm{Im}(f(j))]$ and $\cdot$ denote vector dot product. For any u, the equation $\sum_u f(u)g(u) = 0$ yields $0 = x(u) \cdot \sum_j x(j)g(j) = |x(u)|^2 g(u) + \sum_{j: j \ne u} x(u) \cdot x(j)g(j)$, from which it is clear that there must exist some $v \neq u$ such that $x(u) \cdot x(v) g(v) < 0 \Rightarrow x(u) \cdot x(v) < 0 \Rightarrow \cos\theta < 0$, where θ denotes the angle between vectors $x(u)$ and $x(v)$.

Now, squaring both sides of $|f(u) - f(v)| > |f(u)|$ and rewriting using the fact that $|z|^2 = z\bar{z}$, we have

$$\begin{aligned} 0 &< |f(v)|^2 - (f(u)\overline{f(v)} + f(v)\overline{f(u)}) \\ &= |f(v)|^2 - 2\,\mathrm{Re}(f(u)\overline{f(v)}) \\ &= |f(v)|^2 - 2\,\mathrm{Re}\left(|f(u)||f(v)|e^{i(\varphi_u - \varphi_v)}\right) \\ &= |f(v)|^2 - 2|f(u)||f(v)|\cos\theta; \end{aligned}$$

where in the last step we used Euler's formula and θ denotes the angle between $x(u)$ and $x(v)$. Since $|f(v)|^2 - 2|f(u)||f(v)|\cos\theta > 0$ holds when $\cos(\theta) < 0$, the claimed inequality holds. □

Using the above fact, we now prove Theorem 15.

Proof of Theorem 15. Let ϕ denote the Perron vector (scaled so its entries sum to 1) of the probability transition matrix P of G. Then, from [16], we have

$$\lambda_1 = \inf_{\substack{f \in \mathbb{C}^n \\ \sum_u f(u)\phi(u) = 0}} \frac{\sum_{u \to v} |f(u) - f(v)|^2 \phi(u) P(u, v)}{2 \sum_v |f(v)|^2 \phi(v)}. \tag{2.21}$$

Let f be the harmonic eigenvector achieving λ_1 in Equation (2.21). Let u_0 denote a vertex with $|f(u_0)| = \max_u |f(u)|$. From Fact 1, there must exist v_0 such that $|f(u_0) - f(v_0)|^2 > |f(u_0)|^2$. Let S denote the shortest directed path from u_0 to v_0. By Cauchy-Schwarz, we have:

$$|f(u_0) - f(v_0)|^2 = \left(\sum_{(u,v) \in S} (f(u) - f(v)) \cdot 1 \right)^2 \leq D \sum_{(u,v) \in S} |f(u) - f(v)|^2.$$

Applying this fact, we obtain

$$\lambda_1 = \frac{\sum_{u \to v} |f(u) - f(v)|^2 \phi(u) P(u, v)}{2 \sum_v |f(v)|^2 \phi(v)}$$

$$\frac{\sum_{(u,v)\in S} |f(u)-f(v)|^2\,\phi(u)P(u,v)}{2\,|f(u_0)|^2} \ge \frac{\min_u \phi(u)}{2\max_u d^+(u)}\cdot\frac{\sum_{(u,v)\in S}|f(u)-f(v)|^2}{|f(u_0)|^2}$$
$$\ge \frac{\min_u \phi(u)}{2\max_u d^+(u)}\cdot\frac{\frac{1}{D}|f(u_0)-f(v_0)|^2}{|f(u_0)|^2}$$
$$> \frac{\min_u \phi(u)}{2D\max_u d^+(u)}.$$

□

As an immediate consequence of this theorem, we can apply our previous upper bound the principal ratio to get an absolute lower bound on λ_1 in terms of n, the number of vertices.

Corollary 1. For a strongly connected n-vertex directed graph G with normalized Laplacian eigenvalues $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$, we have

$$\lambda_1 > \frac{1}{\left(\frac{4}{3}+o(1)\right)(n-1)^3\,(n-1)!}.$$

Proof. Apply the bound on the principal ratio in Theorem 10 with $\max_u \phi(u) \le \frac{1-\min_u \phi(u)}{n-1}$ to obtain a lower bound on $\min_u \phi(u)$; then apply this to Theorem 15 with $D, \max_u d^+(u) \le n-1$. □

To compare the above corollary with the undirected case, we can derive a corollary from Theorem 13 in a similar way. Namely, taking diameter $D = n-1$ and $\mathrm{vol}(G) = 2\binom{n}{2}$, we obtain

$$\lambda_1 \ge \frac{1}{n(n-1)^2},$$

for undirected graphs.

2.4.2 A construction with small second eigenvalue

To examine the sharpness of the bound in Theorem 15 and subsequent Corollary 1, we give a construction on n vertices with $\lambda_1 < f(n)$, where

$$f(n) := \frac{2}{(e-1)\left(\frac{n}{2}-1\right)!}.$$

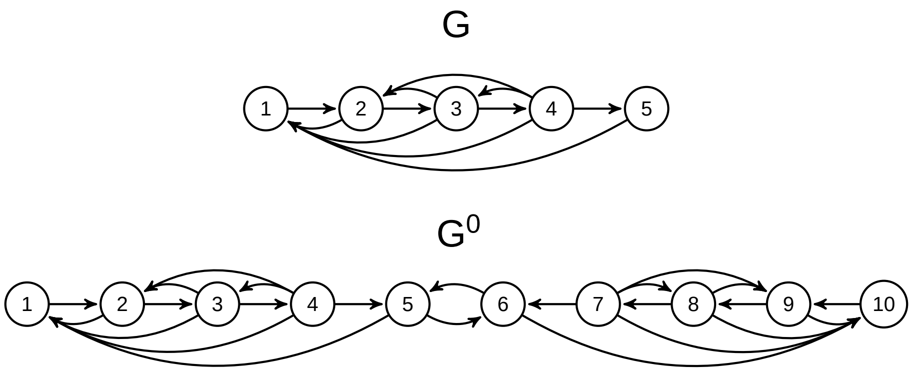

Figure 2.6 : Constructions G and G' in Example 6 for $N = 5$.

We describe the construction below. This construction utilizes the principal ratio extremal graph from Theorem 10 by (loosely speaking) appending a reflected copy of this graph to itself.

Example 6. Let G have vertex set $\{v_1, \ldots, v_N\}$ and edge set given by

$$E(G) = \{(v_i, v_{i+1}) : 1 \le i \le N-1\} \cup \{(v_j, v_i) : 1 \le i < j \le N-1\} \cup \{(v_N, v_1)\}.$$

We define a graph G' on vertex set $\{v'_1, \ldots, v'_N, v'_{N+1}, \ldots, v'_{2N}\}$ that consists of a copy of G on $\{v'_1, \ldots, v'_N\}$ connected to a "reflected" copy of G on $\{v'_{N+1}, \ldots, v'_{2N}\}$ by edges (v'_N, v'_{N+1}) and (v'_{N+1}, v'_N). More precisely,

$$E(G') = \{(v'_i, v'_j) : (v_i, v_j) \in E(G)\} \cup \{(v'_{2N-i+1}, v'_{2N-j+1}) : (v'_i, v'_j) \in E(G)\} \cup \{(v'_N, v'_{N+1}), (v'_{N+1}, v'_N)\}.$$

An illustration for $N = 5$ is shown in Figure 2.6.

We first make some useful observations about the Perron vector of G' and G which will be helpful in proving an upper bound for λ_1 for G'.

Lemma 2. Let $\pi_{G'}$ and π_G denote the Perron vectors of graphs G' and G described in Example 6. Then $\pi_{G'}(v_N) < \pi_G(v_N)$ and $\pi_G(v_N) \le \pi_{G'}(v_N)$.

Proof. Let P and P' denote the probability transition matrices of G and G' respectively. Furthermore, let $x = (x_1, x_2, \ldots, x_N)$ and $x' = (x'_1, x'_2, \ldots, x'_N)$ be such that $xP = x$ and $x'P' = x'$. For $i = 2, \ldots, N-1$, we have identical equations for x_i and x'_i, namely

$$x'_i = \sum_{\substack{i-1 \le j \le N-1 \\ j \ne i}} \frac{x'_j}{j};$$
$$x_i = \sum_{\substack{i-1 \le j \le N-1 \\ j \ne i}} \frac{x_j}{j}.$$

For $i = 1, N$ we have

$$x'_1 = \sum_{2 \le j \le N-1} \frac{x'_i}{i} + \frac{x'_N}{2};$$
$$x_1 = \sum_{2 \le j \le N-1} \frac{x_i}{i} + x_N;$$
$$x'_N = \frac{2x'_{N-1}}{N-1};$$
$$x_N = \frac{x_{N-1}}{N-1}.$$

Substituting the above expressions for x'_N and x_N into the equations for x'_1 and x_1 respectively yields the identical equations

$$x'_1 = \sum_{2 \le j \le N-1} \frac{x'_i}{i} + \frac{x'_{N-1}}{N-1};$$
$$x_1 = \sum_{2 \le j \le N-1} \frac{x_i}{i} + \frac{x_{N-1}}{N-1}.$$

Hence $x'_i = x_i$ for $i = 1, \ldots, N-1$ and thus $x'_N = 2x_N$. Furthermore, by the symmetry of G', it is clear that for $i = 1, \ldots, N$, we have $x'_i = x'_{2N-i+1}$ and thus $\sum_{i=1}^{N} x'_i = \sum_{i=N+1}^{2N} x'_i$. Putting these facts together

$$\pi_{G'}(v'_N) = \frac{x'_N}{\sum_{i=1}^{2N} x'_i} = \frac{2x_N}{2\sum_{i=1}^{N} x'_i} = \frac{x_N}{\sum_{i=1}^{N-1} x_i + 2x_N};$$
$$\pi_G(v_N) = \frac{x_N}{\sum_{i=1}^{N} x_i} = \frac{x_N}{\sum_{i=1}^{N-1} x_i + x_N};$$

from which we obtain $\pi_{G'}(v_N) < \pi_G(v_N)$. Furthermore, as $\pi_G(v_N) = \frac{\pi_G(v_{N-1})}{N-1}$ where $\pi_G(v_{N-1}) < 1$, it is clear that $\lim_{N \to \infty} \pi_G(v_N) = 0$. And the above expressions for $\pi_{G'}(v_N)$ and $\pi_G(v_N)$ imply

$$\pi_G(v_N) + 1 = \frac{\pi_G(v_N)}{\pi_{G'}(v'_N)};$$

from which we can see that $\pi_G(v_N) \leq \pi_{G'}(v'_N)$. □

Next, we obtain the asymptotic behavior of the minimal coordinate of the Perron of G in Example 6.

Lemma 3. Let G be as described in Example 6. Then

$$\pi_G(v_N) \sim \frac{1}{(e-1)\,(N-1)!}.$$

Proof. In [1], it was shown that

$$\pi_G(v_N) = \frac{1}{1+\sum_{k=1}^{N-1}\frac{k^2}{(k+1)!}\sum_{i=k-1}^{N-1} i!}.$$

We claim

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N-1}\frac{k^2}{(k+1)!}\sum_{i=k-1}^{N-1} i!}{(e-1)\,(N-1)!} = 1.$$

Note that

$$\sum_{k=1}^{\infty}\frac{k^2}{(k+1)!} = e-1. \tag{2.22}$$

Reversing the order of summation, we can rewrite

$$\sum_{k=1}^{N-1}\frac{k^2}{(k+1)!}\sum_{i=k-1}^{N-1} i! = \sum_{i=0}^{N-2} i!\left(\sum_{k=1}^{i+1}\frac{k^2}{(k+1)!}\right) + (N-1)!\left(\sum_{k=1}^{N-1}\frac{k^2}{(k+1)!}\right),$$

where by Equation (2.22), the second term above is asymptotic to $(e-1)\,(N-1)!$. Thus, all that remains to be shown is that

$$\lim_{N\to\infty}\frac{\sum_{i=0}^{N-2} i!\sum_{k=1}^{i+1}\frac{k^2}{(k+1)!}}{(e-1)\,(N-1)!} \leq \lim_{N\to\infty}\frac{\sum_{i=0}^{N-2} i!}{(N-1)!} \leq \lim_{N\to\infty}\frac{\frac{5}{3}(N-2)!}{(N-1)!} = 0,$$

where, in the last step, we used the fact that $\sum_{i=0}^{N-2} i! \leq \frac{5}{3}(N-2)!$ holds for $N \geq 3$. □

Using the preceding lemmas, we now prove an upper bound on λ_1 for the construction G^0 described in Example 6. Our main tool will be Theorem 5, the directed Cheeger inequality proved by Chung in [15].

Claim 1. Let directed graphs G and G' be as described in Example 6. Then

$$\lambda_1(G') < 2\phi_G(v_N) \le \frac{2}{(e-1)\cdot(N-1)!},$$

where $\lambda_1(G')$ denotes the first non-trivial eigenvalue of the normalized Laplacian of G' and $\phi_G(v_N)$ denotes the value of the Perron vector of G on vertex v_N.

Proof. Taking $S = \{v'_1, \ldots, v'_N\}$, we have that the Cheeger constant $h(G')$ of G' satisfies

$$\begin{aligned} h(G') &\le \frac{F(\partial S)}{\min\{F(S), F(\bar{S})\}} \\ &= \frac{\phi_{G'}(v'_N)/2}{\sum_{i=1}^{N} \phi_{G'}(v'_i)} \\ &= \phi_{G'}(v'_N) \\ &< \phi_G(v_N), \end{aligned}$$

where, in the last step, we applied Lemma 2 and in the second-to-last step, we applied the fact that $\phi_{G'}(v'_i) = \phi_{G'}(v'_{2N-i+1})$ for $i = 1, \ldots, N$. Applying the upper bound of the Cheeger inequality (Theorem 5) and Lemma 3 yields the result.

□

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Chapter 3

Graphs with many strong orientations

3.1 Introduction

In this chapter, we answer a question posed by Fan Chung related to counting strong orientations: for which possibly sparse and irregular graphs are "most" orientations strongly connected? More precisely, for any given $\epsilon > 0$, we wish to establish minimally restrictive conditions on G so that, with probability $1 - \epsilon$, a random orientation of G is strongly connected, provided the number of vertices n is sufficiently large.

In particular, we show that if a general graph G satisfies a mild eigenvalue condition and mild minimum degree requirement, then a random orientation will be strongly connected with high probability. In fact, we actually prove two main theorems: a "weak" and "strong" version. Both theorems contain an identical minimum degree requirement, but the strong version stipulates a bound on the Cheeger constant (and hence, via the Cheeger inequality, the first nontrivial eigenvalue of the normalized Laplacian, λ_1), whereas the weak form replaces this with a more restrictive condition on the spectral gap of the normalized Laplacian, $\sigma = \max\{|1 - \lambda_1|, |\lambda_{n-1} - 1|\}$. Nonetheless, as the methods used to prove each result are completely different, we include both theorems here.

3.2 Main theorem: strong form

We begin with the stronger form of our main theorem:

Theorem 16. Given any $\epsilon > 0$ and $\alpha > 4$, there exists an integer $N_0 = N_0(\epsilon; \alpha)$ such that for $n \geq N_0$, if G is an n-vertex graph with minimum degree $\delta(G) \geq (1+\epsilon)\log_2 n$ and Cheeger constant $\Phi(G) > \alpha \frac{\log_2 \log_2 n}{\log_2 n}$, then a random orientation of G is strongly connected with probability at least $1 - \frac{1+4\alpha \log_2 n}{n^{\epsilon} \log_2 n}$.

Thus, a graph G satisfying the conditions of Theorem 16 has $(1-o(1))2^{e(G)}$ many strong orientations, where $e(G)$ denotes the number of edges of G. We remind the reader that the Cheeger constant of a graph measures the fewest number of edges leaving a vertex set relative to the "size" of that set. Beyond the bound on the Cheeger constant and the minimum degree requirement, we do not assume the graph necessarily satisfies additional structural properties; in particular, the graph is not assumed to be regular. Not assuming regularity increases the utility of the result, but introduces additional subtleties in the proof, particularly with regard to enumerating connected k sets of the graph.

As we will show in Section 3.2.1, the minimum degree requirement is tight while the bound on the Cheeger constant is tight up to a $\log_2 \log_2 n$ factor. Since the normalized Laplacian eigenvalues of a general graph can be more efficiently computed than its Cheeger constant, it may be useful to reformulate the second condition in Theorem 16 as a spectral condition via the Cheeger inequality.

Corollary 2. In Theorem 16, the condition $\Phi(G) > \alpha \frac{\log_2 \log_2 n}{\log_2 n}$ may be replaced with

$$\frac{\lambda_1(G)}{2} > \alpha \frac{\log_2 \log_2 n}{\log_2 n};$$

where $\lambda_1(G)$ denotes the second eigenvalue of the normalized Laplacian of G.

Here, we emphasize that the spectral condition in Corollary 2 only pertains to the second eigenvalue and thus makes no additional assumptions about the spectral gap $\lambda = \max_{i \geq 1} |1 - \lambda_i|$, which is the key parameter in controlling the discrepancy of a graph. Thus, while we assume a bound on $|1 - \lambda_1|$, we do not assume an additional bound on the other end of the spectrum, $|1 - \lambda_{n-1}|$, beyond the trivial bound that holds for any graph, $\lambda_{n-1} \leq 2$.

A consequence of the Cheeger inequality, by which Corollary 2 follows immediately from Theorem 16, is that for any set $X \subseteq V(G)$ with $\mathrm{vol}(X) \leq \frac{1}{2}\mathrm{vol}(G)$,

$$e(X,\overline{X}) \geq \frac{1}{2}\mathrm{vol}(X). \tag{3.1}$$

As an aside, this uses only the bound $2 \quad \lambda_1$, so we are not using the full strength of Cheeger's inequality. Indeed, on graphs the lower bound $\lambda_1 \quad 2$ is easily proven (for instance, see Lemma 2.1 in [14]). In the Riemannian manifold case, Cheeger's inequality only refers to the lower bound on λ_1 in terms of – the upper bound on λ_1 in terms of is Buser's inequality [10]. Nonetheless, we stick with the convention in graph theory and refer to (3.1) as following from Cheeger's inequality.

Next, in Section 3.2.1 we briefly discuss the isoperimetric condition and minimum degree requirement in Theorem 16. In Section 3.2.2, we present the proof of Theorem 16.

3.2.1 Sharpness of conditions

Before we proceed with the proof of Theorem 16, we briefly discuss the minimum degree requirement and Cheeger constant bound. First, we show that each of these conditions, taken on their own, do not ensure that a random orientation of a graph yields a strongly connected directed graph with any nonzero limiting probability. For instance, Figure 3.1 illustrates the so-called barbell graph on n vertices, which has minimum degree a factor of n but possesses a bridge. Similarly, the graph obtained by connecting a single vertex to exactly one vertex of K_{n-1} has Cheeger constant always at least $1/2$ (as we prove below) but again contains a bridge. Thus, neither condition in Theorem 16, on its own, is sufficient in ensuring the result.

Proposition 15. Let $G = (V, E)$ be the graph on $\{v_1, \ldots, v_n\}$ obtained by connecting a single vertex to exactly one vertex of K_{n-1}. Then $h(G) \geq \frac{1}{2}$ for $n \geq 4$.

Proof. Let $\{v_1, \ldots, v_{n-1}\}$ denote the set of vertices in the clique, and $\{v_1, v_n\}$ the appended edge. For ease of exposition, we assume n is even. For any subset S with $|S| = k \geq 2$ satisfying $\mathrm{vol}(S) \leq \frac{1}{2}\mathrm{vol}(G)$, there are four cases:

$$h(S) = \begin{cases} \dfrac{k(n-k)+1}{k(n-1)+1} & \text{if } v_{n+1} \notin S;\ v_1 \in S; \\[2ex] \dfrac{(k-1)(n-k+1)+1}{(k-1)(n-1)+1} & \text{if } v_{n+1} \in S;\ v_1 \notin S; \\[2ex] \dfrac{(k-1)(n-k+1)}{(k-1)(n-1)+2} & \text{if } v_{n+1} \in S;\ v_1 \in S; \\[2ex] \dfrac{n-k}{n-1} & \text{if } v_{n+1} \notin S;\ v_1 \notin S. \end{cases}$$

Note that $\mathrm{vol}(S) \leq \frac{1}{2}\mathrm{vol}(G) \implies k \leq \frac{n}{2}$, except in Case 2 above, where $k \leq \frac{n}{2} + 1$. However, we may assume $k \leq \frac{n}{2}$, as a straightforward computation shows that when $k = \frac{n}{2}$, the Cheeger constant of a subset in Case 4 is still smaller than for that of a subset in Case 2 when $k = \frac{n}{2} + 1$. We can further restrict attention to Cases 3 and 4 above since the Cheeger constant of such subsets is always smaller than those in Cases 1 and 2, respectively.

Now, for fixed n, the Cheeger constant in Cases 3 and 4 is strictly monotonically decreasing for $2 \leq k \leq \frac{n}{2}$. And whenever $n \geq 4$ and $k \geq 3$, $\frac{(k-1)(n-k+1)}{(k-1)(n-1)+2} > \frac{n-k}{n-1}$. Thus

$$h(G) = \begin{cases} \frac{3}{5} & \text{if } n = 4; \\ \frac{1}{2} + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases}$$

□

Next, we show our minimum degree requirement is sharp while the bound on the Cheeger constant is sharp up to a $\log_2 \log_2 n$ factor. In order to do this, we will make use of the fact that if G is a random d-regular graph, for $d = c \log_2 n$, then G has a Cheeger constant bounded away from zero. Such results were known for fixed d dating to the work of Bollobás [7].

For non-constant degree, as in our case, the easiest approach to such a result is to appeal to the spectra. The study of spectra of random regular graphs has a long history, culminating most famously in Friedman's proof of Alon's second

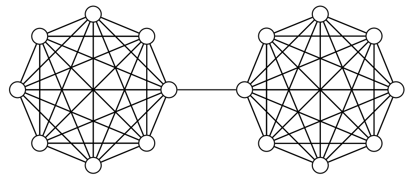

Figure 3.1 : Two copies of $K_{n/2}$ connected by an edge.

eigenvalue conjecture [24]: random regular graphs of xed degree d have second eigenvalue of the adjacency matrix $2\sqrt{d \quad 1}+$ for any > 0, with high probability. This, again unfortunately for our work, focuses on the case with constant degree. Fortunately for our purposes, Broder, Frieze, Suen and Upfal [9] showed that the technique used by Kahn and Szemeredi in [28] works in the case that $d = o(n^{1/2})$, and shows that the second eigenvalue of the adjacency matrix is $O(\sqrt{d})$ for such graphs. In terms of normalized Laplacian eigenvalues, this shows that $_1 \quad 1 \quad O(d^{1/2})$ in this regime, and through Cheeger's inequality random d-regular graphs have Cheeger constant satisfying $> \frac{1}{4}$ with high probability so long as n is su ciently large. We mention that this problem is still attracting attention, as just recently, Cook, Goldstein and Johnson [40] proved that the second adjacency eigenvalue for a random d-regular graph is still $O(\sqrt{d})$ for $d = o(n^{2/3})$.

We now use the fact that a $\log_2 n$ regular graph has Cheeger constant at least $1/4$ with high probability when considering the following example, which shows our minimum degree requirement is sharp.

Example 7. Let G^0 be a random t regular graph on $N = 2^t$ vertices.

Proposition 16. G^0 has minimum degree $\log_2 N$ and, with high probability, Cheeger constant at least $\frac{1}{4}$. However, a random orientation of G^0 is disconnected with limiting probability at least $1 \quad \frac{1}{e}$.

Proof. We show a random orientation of G^0 is disconnected with limiting probability at least $1 \quad \frac{1}{e}$. Since G^0 is $\log_2 N$ regular, the probability a vertex is a sink in a random orientation is $\frac{1}{N}$. Assume the vertices are labeled and let B_i denote the event that vertex i is a sink. For xed k, de ne

$$S^{(k)} = \sum_{\{i_1,\ldots,i_k\} \in \binom{V(G')}{k}} P(B_{i_1} \cap \cdots \cap B_{i_k}).$$

By Brun's sieve [5, Theorem 8.3.1], if we show that for every fixed k

$$\lim_{N \to \infty} S^{(k)} = \frac{1}{k!},$$

then the limiting probability there are no sinks in a random orientation of G' is $\frac{1}{e}$. Note that if $\{i, j\} \in E(G')$, then $P(B_i \cap B_j) = 0$. Thus, we may rewrite the sum for $S^{(k)}$ as over all independent sets with k vertices. Accordingly, since we need each of the $t = \log_2 N$ edges for each of the k vertices to be oriented so that each is a sink, $P(B_{i_1} \cap \cdots \cap B_{i_k}) = \left(\frac{1}{2^t}\right)^k = \frac{1}{N^k}$. At most, every k-subset of $V(G')$ is an independent set, yielding the upper bound

$$S^{(k)} \le \binom{N}{k} \frac{1}{N^k} \le \frac{1}{k!},$$

and at least, there are $\frac{1}{k!} \cdot N(N - \log_2 N) \cdots (N - (k-1)\log_2 N) \ge \frac{(N - k\log_2 N)^k}{k!\, N^k}$ independent sets of size k, yielding the lower bound

$$S^{(k)} \ge \frac{\left(1 - \frac{k \log_2 N}{N}\right)^k N^k}{k!\, N^k} \to \frac{1}{k!}.$$

□

Having shown that the minimum degree condition in Theorem 16 is sharp, we now use G' to construct a graph G to show the Cheeger constant condition in Theorem 16 is sharp up to a $\log_2 \log_2 n$ factor.

Example 8. For any integer $c > 1$, consider the graph G on n vertices obtained from G' by appending to each vertex in G' vertex disjoint complete graphs on ct vertices. (Equivalently, G is constructed by taking N vertex disjoint cliques on ct vertices, selecting from each of them a distinguished vertex, and amongst the N distinguished vertices, placing a t regular random graph). See Figure 3.2 for one example of this construction.

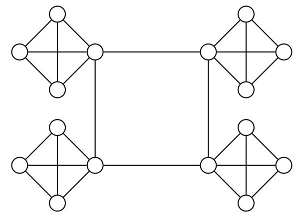

Figure 3.2 : The graph G in Example 8 with $t = c = 2$.

Proposition 17. G has minimum degree $\Theta(\log_2 n)$ and Cheeger constant $\Phi(G) = \Theta(\log_2^{-1} n)$. However, a random orientation of G is disconnected with limiting probability at least $1 - \frac{1}{e}$.

Proof. First, recalling that G is constructed by appending disjoint complete graphs to each vertex in G^0, Proposition 16 immediately implies a random orientation of G is disconnected with limiting probability at least $1 - \frac{1}{e}$. Next, we examine the minimum degree and Cheeger constant of G. Note that the graph G is on $n = ctN$ vertices, and $\log_2 n = t + \log_2(ct)$. For t large enough, the minimum degree in the graph (which is $ct - 1$) is at least $\frac{c \log_2 n}{2}$ and the maximum degree is $(c+1)t - 1 < 2c \log_2 n$. For any subset $X \subseteq V(G)$ with $\mathrm{vol}(X) < \mathrm{vol}(G)/2$, we will show that

$$\frac{e(X; \bar{X})}{\mathrm{vol}(X)} = \Theta(\log_2^{-1} n):$$

Note that since every vertex has degree $\Theta(\log_2 n)$ it suffices to show that for all subsets of cardinality at most $\frac{n}{2}$,

$$\frac{e(X; \bar{X})}{|X|} = \Theta(1) \ :$$

This is what we shall do. Let $S \subseteq V(G)$ denote the vertices of G^0 (contained as a subgraph of G.) Let $S_1; \ldots; S_N$ denote the vertices contained (respectively) in each of the N cliques. Note $|S_i \cap S| = 1$ for all i, as there is a unique distinguished vertex per clique. Fix a set $X \subseteq V(G)$. Define the sets

$$S^0 = X \cap S;$$

$$T_1 = \{x \in X : x \in S_i \text{ with } (S_i \cap S^0) \neq \emptyset \text{ for some } i \in [N]\};$$

$$T_2 = \{x \in X : x \in S_i \text{ with } (S_i \cap S^0) = \emptyset \text{ for some } i \in [N]\}:$$

Note that $S^0 \quad T_1$ and T_1 and T_2 partition X. We observe that

$$\frac{e(X;X)}{|X|} = \frac{e(T_1;X) + e(T_2;X)}{|T_1| + |T_2|}:$$

By the real number inequality

$$\frac{a+b}{c+d} \quad \min \quad \frac{a}{c}; \frac{b}{d} \quad ;$$

valid for positive $a; b; c; d$ it su ces to show that both $\frac{e(T_1;X)}{|T_1|}$ and $\frac{e(T_2;X)}{|T_2|}$ are both (1) (unless one of them is $\frac{0}{0}$ { note that both of them cannot be since X is non-empty).

We begin by proving that $\frac{e(T_2;X)}{|T_2|} = (1)$ so long as T_2 is non-empty. Let $r_i = |S_i \cap T_2|$. Note that $r_i \quad ct \quad 1$ for every i, as the distinguished vertices are not in T_2. Further note that since S_i is a clique, the r_i vertices in S_i are adjacent to all remaining $ct \quad r_i$ vertices in the clique which are in X. Thus

$$e(T_2;X) = \sum_i r_i(ct \quad r_i) \quad \sum_i r_i = |T_2|;$$

so $e(T_2;X)/|T_2| = (1)$.

It is slightly more complicated to bound $\frac{e(T_1;X)}{|T_1|}$. Similarly, we let $n_i = |S_i \cap T_1|$. Let $m = |T_1 \cap S|$. Then

$$e(T_1;X) = e(T_1 \cap S; X \cap S) + \sum_i n_i(ct \quad n_i):$$

Since G^0 has $(G^0) \quad \frac{1}{4}$,

$$e(T_1 \cap S; X \cap S) \quad \frac{1}{4}\min\{m; N \quad m\} \quad \log_2 N:$$

If $m \quad \frac{9N}{10}$ this is su cient to show $\frac{e(T_1;X)}{|T_1|} = (1)$, since $|T_1| = O(m \log_2 N)$ and $e(T_1 \cap S; X \cap S) = (\quad m \log_2 N)$. Otherwise, if $m > \frac{9N}{10}$, without loss of generality $n_1; n_2; \ldots; n_m$ are positive. Consider the function:

$$f(n_1; n_2; n_3; \ldots; n_m) = \sum_i n_i(ct \quad n_i):$$

Note that if $x \leq y$,

$$(x+1)(ct-(x+1))+(y-1)(ct-(y-1))-(x(ct-x)+y(ct-y))=2(y-x-1)<0.$$

Thus, for any two arguments of the function f, increasing the larger by 1 while decreasing the smaller by 1 decreases the function. Since f is symmetric in its variables, we may relabel them so that $n_1 \geq \cdots \geq n_m$ and repeatedly apply the above observation to yield:

$$f(n_1; n_2; \ldots; n_m) \geq f(ct; ct; \cdots; ct; \cdot; 1; 1; \ldots; 1; 1);$$

so that the arguments sum to $\sum n_i$ and $1 \leq \cdot \leq ct$. Since $\sum n_i \geq \frac{n}{2}$ and $m > \frac{9N}{10}$, this means that there are at least $\frac{4N}{10}$ 1's, so

$$f(n_1; n_2; n_3; \ldots; n_m) \geq f(ct; ct; \cdots; ct; \cdot; 1; 1; \ldots; 1; 1) \geq \frac{4N}{10} \cdot 1(ct-1) = \Omega(n).$$

This shows $e(T_1; X) = \Omega(n)$, hence $\frac{e(T_1;X)}{|T_1|} = \Omega(1)$. Thus we have shown $\Phi(G) = \Omega(\log_2^{-1} n)$. □

3.2.2 Proof of Theorem 16

Our general approach to proving Theorem 16 is based on the observation that a directed graph is strongly connected if and only if every nonempty proper subset $X \subsetneq V(G)$ has an edge both entering and leaving it. Namely, we bound the probability that every connected set $X \subseteq V(G)$ with $\mathrm{vol}(X) \leq \mathrm{vol}(G)/2$ has an edge both entering and leaving it.

Definition 6. For a subset X of vertices let B_X be the event that $\mathrm{vol}(X) \leq \mathrm{vol}(G)/2$, X is connected in G and X has either no edges oriented into it or out of it. Note only the third property here is random – if X does not have one of the first two properties, $P(B_X) = 0$ deterministically. We further define

$$B_k = \bigcup_{\substack{X \subseteq V(G) \\ |X| = k}} B_X.$$

We estimate $P(\bigcup_k B_k) \leq \sum_k P(B_k)$ by dividing k into two regimes. First we prove that every small subset (where $|X| \leq c\log_2 n$) has an edge entering and leaving:

Regime 1: We claim $\sum_{k=1}^{\delta = 2\log_2 n} P(B_k) < \frac{4}{n}$.

Proof. We begin by noting that for a given set X of size k, there are at most $\binom{k}{2}$ edges induced on X and hence, recalling that δ denotes the minimum degree, there are at least $k\delta - \binom{k}{2}$ edges leaving. Note that in this regime, $k\delta - \binom{k}{2} > 0$ since $k \leq \frac{\delta}{2}\log_2 n$. For a given set X,

$$P(B_X) \leq 2^{-k\delta + \binom{k}{2} + 1};$$

and this gives an estimate

$$P(B_k) \leq \binom{n}{k} 2^{-k\delta + \binom{k}{2} + 1} =: b_k:$$

We note that if $k \leq \frac{\delta}{2}\log_2 n$,

$$\begin{aligned} \frac{b_{k+1}}{b_k} &= \frac{\binom{n}{k+1} 2^{-(k+1)\delta + \binom{k+1}{2} + 1}}{\binom{n}{k} 2^{-k\delta + \binom{k}{2} + 1}} \\ &= \frac{(n-k)2^k}{(k+1)2^\delta} \\ &\leq \frac{2^k}{n} \leq \frac{1}{2}: \end{aligned}$$

Then

$$\sum_{k=1}^{\frac{\delta}{2}\log_2 n} P(B_k) \leq 2P(B_1) \leq 2n2^{1-(1+\epsilon)\log_2 n} = 4n^{-\epsilon}:$$

□

Regime 2: We claim $\sum_{k=\frac{\delta}{2}\log_2 n}^{n} P(B_k) \leq \frac{1}{n - \log_2 n}$.

Proof. For large sets, we must take greater care { the number of edges that could be induced in sets is much larger, we utilize our lower bound on the Cheeger constant to ensure many edges leave each set. Since the number of potential k sets grows large as well, we will restrict attention to counting only connected sets so as to not over count.

To this end, we will enumerate connected k sets by considering rooted spanning trees in G, which we will consider labeled. The shape of spanning trees, can of course, vary wildly. For the purposes of this work we will enumerate them by their exposure sequence.

Definition 7. An exposure sequence $= ({}_1; {}_2; \ldots; {}_{k-1})$ of a labeled rooted spanning tree on k vertices is determined as follows: newly label the vertices in breadth-first order, with ties broken by the original labeling of the tree. Then, under this new labeling ${}_i$ is the number of children of vertex v_i in the tree. See Figure 3.3 for an example.

Therefore, an exposure sequence of a rooted spanning tree on k vertices is an (ordered) list of $(k-1)$ non-negative integers $({}_1; {}_2; \ldots; {}_{k-1})$ satisfying $\sum_{i \le j} {}_i \ge j$ and $\sum_{i=1}^{k-1} {}_i = k-1$. A given exposure sequence of $k-1$ numbers uniquely determines the shape of the rooted, spanning tree on k vertices. Since these vertices are labeled in breadth-first order, the kth vertex is necessarily a leaf of the tree, so by convention we have ${}_k = 0$. We note that an exposure sequence for a rooted spanning tree on k vertices can be thought of as a staircase walk on the square lattice from $(0, 0)$ to $(k-1; k-1)$ which never crosses the diagonal. Namely, the staircase walk corresponding to exposure sequence is formed by taking ${}_i$ steps east and 1 step north for $i = 1; \ldots; k-1$ (see Figure 3.3). Thus, counting all possible exposure sequences is equivalent to counting all Dyck paths on the square lattice, which is given by the Catalan numbers $c_{k-1} = \frac{1}{k}\binom{2(k-1)}{k-1}$.

We will enumerate all of the rooted subtrees in G on k vertices by their exposure sequence. Our task now is to bound the following sum:

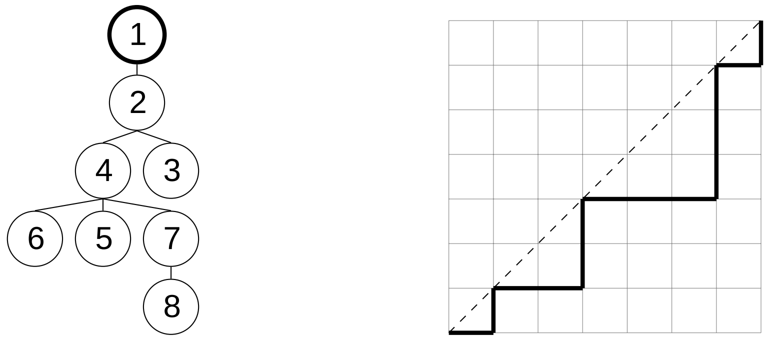

Figure 3.3 : Left: a breadth-first vertex labeling of a rooted tree yielding exposure sequence $\alpha = (1, 2, 0, 3, 0, 0, 1)$. Right: the staircase walk corresponding to exposure sequence $\alpha = (1, 2, 0, 3, 0, 0, 1)$.

$$P(B_k) \le \sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_{k-1})} \sum_{v_1 \in V(G)} \sum_{\{v_2,v_3,\ldots,v_{1+\alpha_1}\} \in \binom{N(v_1)}{\alpha_1}} \sum_{\{v_{2+\alpha_1}, v_{3+\alpha_1}, \ldots, v_{1+\alpha_1+\alpha_2}\} \in \binom{N(v_2)}{\alpha_2}} \cdots \sum_{\{v_{k-\alpha_{k-1}+1},\ldots,v_k\} \in \binom{N(v_{k-1})}{\alpha_{k-1}}} P(B_X); \quad (3.2)$$

where $X = \{v_1, \ldots, v_k\}$ and $\binom{N(v_i)}{\alpha_i}$ denotes the set of all sets of α_i vertices adjacent to v_i in the original graph G. For any X which is connected in the original graph,

$$P(B_X) = \begin{cases} 0 & \text{if } \mathrm{vol}(X) > \frac{\mathrm{vol}(G)}{2} \\ 2^{1-e(X,\bar{X})} & \text{if } \mathrm{vol}(X) \le \frac{\mathrm{vol}(G)}{2} \end{cases}.$$

In the second case,

$$P(B_X) \le 2^{1-e(X,\bar{X})} \le 2^{1-\epsilon\,\mathrm{vol}(X)} = 2^{1-\epsilon\sum_{v_i \in X} \deg(v_i)}.$$

Since this bounds $P(B_X)$ above by a positive quantity (and $P(B_X)$ is otherwise zero), the inequality

$$P(B_X) \le 2^{1-\epsilon\sum_{v_i \in X} \deg(v_i)}; \quad (3.3)$$

holds for every X. We now use (3.3) to bound the right hand side of (3.2). We wish to collapse a term of the form

$$\sum_{\binom{N(v_j)}{\alpha_j}} 2^{1-\epsilon\deg(v_j)};$$

as (after having already bounded each of the summands for v_i where $i > j$) we will have ensured that the summand is independent of the $_j$ vertices chosen. Thus,

$$\sum_{\binom{N(v_j)}{j}} 2^{1 \quad \deg(\ v_j)} = \binom{\deg(v_j)}{j} 2^{1 \quad \deg(\ v_j)} :$$

We will give an upper bound of this term which is independent of v_j, depending only on $_j$ and , and this will allow us to continue collapsing the sum (3.2). We nd three di erent upper bounds for this term for the cases when $_j = 0$, $_j = 1$, or $_j > 1$.

Case 1: $_j = 0$.

If v_j is a leaf of the embedded spanning tree (which corresponds to $_j = 0$), we simply bound

$$2^{1 \quad \deg(\ v_j)} \quad 2^{1 \quad (1+ \ )\log_2 n} :$$

Case 2: $_j > 1$.

Since the terms we are interested in have the general form $\binom{\deg(v_j)}{j} 2^{1 \quad \deg(\ v_j)}$, we investigate the associated sequence de ned by xing $_j$ and varying $\deg(v_j)$. In general, let

$$_{s;t} = \binom{s}{t} 2^{1 \quad s} ;$$

so that the terms appearing above are $_{\deg(v_j);\ j}$. Then for a xed t and varying s, the sequence $_{s;t}$ is unimodal. We have that

$$\frac{_{s;t}}{_{s+1;t}} = 1 \quad \frac{t}{s+1} \quad 2 :$$

Thus the maximum of $_{s;t}$, for a xed t, is achieved by the smallest s such that $_{s+1;t} < \ _{s;t}$, yielding

$$\frac{t}{s+1} < 1 \quad 2 \quad ;$$

or equivalently

$$(s+1) > \frac{t}{1 \quad 2} :$$

Thus the maximum of $_{s;t}$ occurs when

$$s_{max(t)} = \frac{t}{1 \quad 2} :$$

Indeed, extending the binomial coe cients to the reals in the usual way, the oor function can be dropped. Recalling that ; s 0, for xed t we have:

$$_{s;t} \quad \frac{(1 \quad 2 \quad)^{1}t}{t} \quad 2^{1 \quad (1 \quad 2 \quad)^{1}t} : \qquad (3.4)$$

Next, we use the entropy bound:

$$\frac{n}{k} \quad \frac{n^n}{k^k (n \quad k)^{n \quad k}} = 2^{nH(k=n)};$$

where $H(q) = \quad q \log_2 q \quad (1 \quad q) \log_2 (1 \quad q)$ is the binary entropy function.

Applying this bound to the binomial coe cient in (3.4) yields:

$$\log_2 \frac{(1 \quad 2 \quad)^{1}t}{t} \quad (1 \quad 2 \quad)^{1} t H (1 \quad 2 \quad)$$

$$= \quad (1 \quad 2 \quad)^{1} t \ (1 \quad 2 \quad) \log_2 (1 \quad 2 \quad) + 2 \quad \log_2 (2 \quad)$$

$$= t \quad \log_2 (1 \quad 2 \quad) + \frac{}{2 \quad 1} :$$

Combining this upper bound with (3:4) and simplifying, we have that

$$_{s;t} \quad 2^{1 + t(\quad \log_2 (1 \quad 2 \quad) \quad)} : \qquad (3.5)$$

We will now provide constant upper bounds on the terms involving in the exponent of (3.5). Setting $f(x) = \quad \log_2 (1 \quad 2^{\ x})$, we have $f^0(x) = \frac{1}{2^x \quad 1}$ and so for $x > 0$,

$$f(x) = f(1) + \int_1^x f^0(t)\, dt = 1 + \int_x^1 \frac{1}{2^t \quad 1} dt:$$

Since $1 + x \ln 2 \quad e^{x \ln 2} = 2^x$, we have that for $x > 0$,

$$f(x) \quad 1 + \int_x^1 \frac{1}{t \ln 2} dt = 1 \quad \log_2 (x):$$

We may use this to bound (3.5), yielding

$$_{s;t} < 2^{1 \quad t \log_2 (\ ) + \quad t} :$$

Although we will only apply this when $_j > 1$, this gives the general bound, good for any $\deg(v_j); \; _j$ that

$$\deg(v_j) \quad _j \; 2^{1 \quad \deg(v_j)} \quad 2^{1 \quad _j \log_2(\;)+ \quad _j}:$$

Case 3: $_j = 1$.

In this case, the previous bound does not su ce for our purposes. Here, we improve the bound by observing that our conditions imply that $\deg(v_j) > s_{max}(t)$: Indeed, our condition that $\; > \frac{\log_2 \log_2 n}{\log_2 n}$ implies that for n su ciently large, $(1 + \;) \log_2 n > (1 \quad 2 \quad)^{1}$. Hence we are interested in $\deg(v_j); \; _j$ and by the unimodality of the $_{s;t}$ for t xed, we can derive the bound:

$$\begin{aligned} \deg(v_j) \quad _j \; 2^{1 \quad \deg(v_j)} &< (1 + \;) \log_2 n \quad _j \; 2^{1 \quad (1+ \;) \log_2 n} \\ &< ((1 + \;) \log_2 n)^{\;_j} 2^{1 \quad (1+ \;) \log_2 n} \\ &= 2^{1+ \; _j [\log_2(1+ \;)+\log_2 \log_2 n] \quad (1+ \;) \log_2 n}; \end{aligned} \qquad (3.6)$$

which for $_j = 1$ simpli es to

$$\deg(v_j) \quad 1 \; 2^{1 \quad \deg(v_j)} < 2^{1+[\log_2(1+ \;)+\log_2 \log_2 n] \quad (1+ \;) \log_2 n}:$$

Collecting our results from Cases 1, 2, and 3, we have established the following:

$$\sum_{\binom{N(v_j)}{j}} 2^{1 \quad \deg(v_j)} = \deg(v_j) \quad _j \; 2^{1 \quad \deg(v_j)} \quad \begin{cases} 2^{1 \quad (1+ \;) \log_2 n} & \text{if } _j = 0 \\ 2^{1+[\log_2(1+ \;)+\log_2 \log_2 n] \quad (1+ \;) \log_2 n} & \text{if } _j = 1 \\ 2^{1+ \; _j \log_2(1= \;)+ \quad _j} & \text{if } _j > 1 \end{cases} : \qquad (3.7)$$

Before we collapse the sum (3.2) using (3.7), we make a few simple combinatorial observations concerning exposure sequences of rooted spanning trees. Recalling that a degree of a vertex in the spanning tree is $_1$ for v_1, and $_i + 1$ for v_i, we de ne the following:

Definition 8. For an exposure sequence $\sigma = (\sigma_1, \dots, \sigma_{k-1})$, let

$$\ell(\sigma) = 1 + |\{j \le k-1 : \sigma_j = 0\}|$$

denote the number of leaves of the spanning tree described by the sequence and we let

$$p(\sigma) = |\{j \le k-1 : \sigma_j = 1\}|.$$

Lemma 4. For any exposure sequence, we have

$$p(\sigma) + \ell(\sigma) \ge \frac{k}{2};$$
$$\sum_{j : \sigma_j \ge 2} \sigma_j < k - p(\sigma).$$

Proof. For the first observation, note that if $p(\sigma) + \ell(\sigma) < \frac{k}{2}$, then there are at least $\frac{k}{2}$ terms in σ that are at least 2, yielding the contradiction:

$$k = (k/2) \cdot 2 \le \sum_{i=1}^{k-1} \sigma_i = k - 1.$$

And, for the second observation:

$$k - 1 = \sum_{i=1}^{k-1} \sigma_i = \sum_{j : \sigma_j \ge 2} \sigma_j + \sum_{j : \sigma_j = 1} \sigma_j + \sum_{j : \sigma_j = 0} \sigma_j$$
$$= \sum_{j : \sigma_j \ge 2} \sigma_j + p(\sigma).$$

□

We now proceed to bound $P(B_k)$ (3.2). We will take logarithm here for readability so that every term would not appear in the exponent – this should be viewed most naturally by exponentiating both sides. Iteratively applying (3.7), we obtain that for a fixed $\sigma = (\sigma_1, \cdots, \sigma_{k-1})$ and $v_1 \in V(G)$

$$\log_2 P(B_k) \le \log_2 \left(\sum_{\{v_2, v_3, \dots, v_{1+\sigma_1}\} \in \binom{N(v_1)}{\sigma_1}} \; \sum_{\{v_{2+\sigma_1}, v_{3+\sigma_1}, \cdots, v_{1+\sigma_1+\sigma_2}\} \in \binom{N(v_2)}{\sigma_2}} \cdots \sum_{\{v_{k-\sigma_{k-1}+1}, \dots, v_k\} \in \binom{N(v_{k-1})}{\sigma_{k-1}}} P(B_X) \right)$$

$$\left[\sum_{j:\ j\ \ 2} 1 + \ _j \log_2(1=) + \ _j \right] + p( )[1 + \log_2(1 + \ ) + \log_2 \log_2 n]$$

$$(p( ) + `( )) \quad (1 + \ ) \log_2 n + `( )$$

$$= \left[\sum_{j:\ j\ \ 2} \ _j \log_2(1=) + \ _j \right] + p( )[\log_2(1 + \ ) + \log_2 \log_2 n]$$

$$(p( ) + `( )) \quad (1 + \ ) \log_2 n + k:$$

Continuing, we apply Lemma 4 to yield

$$\left[\sum_{j:\ j\ \ 2} \ _j \log_2(1=) + \ _j \right] + p( )[\log_2(1 + \ ) + \log_2 \log_2 n]$$

$$(p( ) + `( )) \quad (1 + \ ) \log_2 n + k$$

$$(k \quad p( ))(\log_2(1=) + 1) + \quad p( )[\log_2(1 + \ ) + \log_2 \log_2 n]$$

$$\frac{k}{2} \quad (1 + \ ) \log_2 n + k:$$

Next, using the fact that $\ > \ \frac{\log_2 \log_2 n}{\log_2 n}$ for some (large) constant , we obtain

$$(k \quad p( ))(\log_2(1=) + 1) + \quad p( )[\log_2(1 + \ ) + \log_2 \log_2 n]$$

$$\frac{k}{2} \quad (1 + \ ) \log_2 n + k$$

$$< (k \quad p( )) \log_2 \log_2 n + (k \quad p( )) + p( ) \log_2 (1 + \ )$$

$$+ p( ) \log_2 \log_2 n \quad \frac{k}{2}(1 + \ ) \log_2 \log_2 n + k$$

$$k \quad \log_2 \log_2 n \ \ 1 \quad \frac{}{2}(1 + \ ) \quad + (2 + \log_2(1 + \ )) \quad : \qquad (3.8)$$

Finally, for $\ > 4$ and n su ciently large we have:

$$k \quad \log_2 \log_2 n \ \ 1 \quad \frac{}{2}(1 + \ ) \quad + (2 + \log_2(1 + \ ))$$

$$k (\log_2 \log_2 n [1 \quad 2(1 + \ )])$$

$$= \quad k \ \frac{\log_2 \log_2 n}{2} \quad 2 \quad ^{1} + 4$$

$$k(2 \quad ^{1} + 4):$$

Therefore, by assuming n and α are large enough, for any fixed $\boldsymbol{\ell}$ and $v_1 \in V(G)$ we have that:

$$\sum_{\{v_2,v_3,\dots,v_{1+\ell_1}\} \in \binom{N(v_1)}{\ell_1}} \; \sum_{\{v_{2+\ell_1},v_{3+\ell_1},\dots,v_{1+\ell_1+\ell_2}\} \in \binom{N(v_2)}{\ell_2}} \cdots \sum_{\{v_{k-\ell_{k-1}+1},\dots,v_k\} \in \binom{N(v_{k-1})}{\ell_{k-1}}} P(B_X) \le 2^{-(2\alpha^{-1}+4)k}.$$

Using the above bound and recalling that there are $C_{k-1} = k^{-1}\binom{2(k-1)}{k-1}$ many exposure sequences, we now bound all of (3.2) as:

$$\begin{aligned}
P(B_k) &\le \sum_{\boldsymbol{\ell}=(\ell_1,\ell_2,\dots,\ell_{k-1})} \sum_{v_1 \in V(G)} \sum_{\{v_2,v_3,\dots,v_{1+\ell_1}\} \in \binom{N(v_1)}{\ell_1}} \\
&\qquad \sum_{\{v_{2+\ell_1},v_{3+\ell_1},\dots,v_{1+\ell_1+\ell_2}\} \in \binom{N(v_2)}{\ell_2}} \cdots \sum_{\{v_{k-\ell_{k-1}+1},\dots,v_k\} \in \binom{N(v_{k-1})}{\ell_{k-1}}} P(B_X) \\
&\le \sum_{\boldsymbol{\ell}=(\ell_1,\ell_2,\dots,\ell_{k-1})} \sum_{v_1 \in V(G)} 2^{-(2\alpha^{-1}+4)k} \\
&\le nk^{-1}\binom{2(k-1)}{k-1} 2^{-(2\alpha^{-1}+4)k} \\
&\le nk^{-1}4^{k-1}2^{-(2\alpha^{-1}+4)k} \\
&= 2^{\log_2 n-\left(\log_2 k+(2\alpha^{-1}+2)k+2\right)} \\
&\le 2^{-(\log_2 k+2k+2)};
\end{aligned}$$

where, in the last inequality, we used that $k \ge \frac{\alpha}{2}\log_2 n$. Thus:

$$\begin{aligned}
\sum_{k=\frac{\alpha}{2}\log_2 n}^{n} P(B_k) &\le \sum_{k=\frac{\alpha}{2}\log_2 n}^{n} 2^{-(\log_2 k+2k+2)} \\
&\le 2 \cdot 2^{-\left(\log_2(\frac{\alpha}{2}\log_2 n)+\alpha\log_2 n+2\right)} \\
&= \frac{1}{\alpha n^{\alpha}\log_2 n}.
\end{aligned}$$

This completes our estimate for Regime 2.

□

Finally, combining the estimates we derived in each regime, we see that

$$P\left(\bigcup_{k=1}^{n} B_k\right) \le \sum_{k=1}^{n} P(B_k) \le \frac{1+4\log_2 n}{n\log_2 n};$$

and thus with probability at least $1 - \frac{1+4\log_2 n}{n\log_2 n} = 1 - o(1)$, a random orientation of G is strongly connected, completing our proof of Theorem 16.

□

3.3 Main theorem: weak form

Here, we present a related, but somewhat weaker form of Theorem 16 that replaces the isoperimetric condition with a more restrictive condition on the spectral gap of the normalized Laplacian, $\lambda = \max\{|1-\lambda_1|, |\lambda_{n-1}-1|\}$.

Theorem 17. There exist $c, c', c'' \in \mathbb{R}^+$ and $n_o \in \mathbb{Z}^+$ such that if G is a graph with $|V(G)| > n_o$, minimum degree $\delta > c\log n$ and spectral gap of the normalized Laplacian $\lambda < c'$, then a randomly oriented copy of G is strongly connected with probability $p \ge 1 - \frac{c''}{n}$.

As before in Theorem 16, we remark that our approach here also does not assume the regularity of G and that each condition on G is insufficient on its own to imply the result. In particular, that the assumption on the spectral gap λ of G is insufficient can be easily seen by considering a $\log_2 n$-regular graph. From the Alon-Bopanna bound [41], one can conclude that the spectral gap the normalized Laplacian satisfies

$$\frac{2\sqrt{\log_2 n - 1}}{\log_2 n} = o(1);$$

while a random orientation fails to be strongly connected with high probability for the same reasons described in Proposition 16. To prove our theorem, we give an algorithm that attempts to construct a directed path between two vertices by simultaneously building the successive out-neighborhoods of one vertex and in-neighborhoods of another vertex. Once each neighborhood has expanded to a suitable size, we bound the number of edges between them. We establish the result by bounding the probability of failure at each step in the process.

3.3.1 Tools and neighborhood expansion algorithm

We adopt the following notation, some of which will be defined more explicitly in the algorithm: for a subset $S \subseteq V(G)$ where G is the underlying undirected host graph, $\mathrm{vol}(S) = \sum_{v \in V(G)} d_v$, $N(S) = \{u : u \sim s \in S\}$, and $e(X; Y)$ denotes the number of (undirected) edges between $X; Y$. When considering the resulting randomly oriented digraph, $U(t)$ denotes the "unexplored" vertices at time t available for inclusion in the subsequent neighborhood expansion. We let $V_x(t)$ and $V_y(t)$ denote the current unexplored out and in-neighborhoods of vertices x and y respectively at time t. At each time step, we write the "smaller" and "larger" neighborhoods as follows:

$$V_{min}(t) = \begin{cases} V_x(t) & \text{if } \mathrm{vol}(V_x(t)) \le \mathrm{vol}(V_y(t)) \\ V_y(t) & \text{otherwise} \end{cases}:$$

Similarly, we define

$$V_{max}(t) = \begin{cases} V_x(t) & \text{if } \mathrm{vol}(V_x(t)) > \mathrm{vol}(V_y(t)) \\ V_y(t) & \text{otherwise} \end{cases}:$$

We use $N^+(V_x(t))$ and $N^-(V_y(t))$ to denote the out and in-neighborhoods of $V_x(t)$ and $V_y(t)$ while $N^{\pm}(V_{min}(t))$ is understood to mean $N^+(V_{min}(t))$ if $V_{min}(t) = V_x(t)$ and $N^-(V_{min}(t))$ otherwise. Our proof utilizes the following two theorems. First, we use the discrepancy inequality, sometimes referred to as the expander mixing lemma by others.

Theorem 18 (Discrepancy Inequality [14]). Let G be a graph with spectral gap $\lambda = \max\{1 - \lambda_1; \lambda_{n-1} - 1\}$ of the normalized Laplacian $\mathcal{L}$. Then

$$\left| e(X; Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \le \lambda \sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)}:$$

The above result can be strengthened to obtain:

$$\left| e(X; Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \le \lambda \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(G)}:$$

As an aside, we mention that Chung and Kenter, as well as Butler, proved several analogs of the above discrepancy inequality for directed graphs. While not relevant for our purposes here, we refer the interested reader to [17] and [11] respectively. Secondly, we utilize the Chernoff bounds [13], a typical statement of which, taken from [18], is the following:

Theorem 19 (Chernoff Bounds [13]). *Let $X_1, \ldots, X_n$ be independent random variables with $P(X_i = 1) = p_i$, $P(X_i = 0) = 1 - p_i$, $X = \sum_{i=1}^n X_i$, and expectation $E(X) = \sum_{i=1}^n p_i$. Then we have the following lower and upper tails*

$$P(X \le E(X) - \lambda) \le e^{-\lambda^2/2E(X)};$$

$$P(X \ge E(X) + \lambda) \le e^{-\frac{\lambda^2}{2(E(X)+\lambda/3)}}.$$

We try to construct a directed path between two vertices algorithmically.

Algorithm 1 Find directed path $x, \ldots, y$

input: vertices x, y in randomly oriented digraph

output: success or failure

```
V_x(0) ← {x}                                    } Step 0
V_y(0) ← {y}
U(0) ← V(G) \ {x, y}

V_x(1) ← N^+(x)                                 } Step 1
V_y(1) ← N^-(y)
U(1) ← V(G) \ {x, y}
if V_x(1) ∩ V_y(1) ≠ ∅ or y ∈ V_x(1) then
    return success
end if
if |V_x(1)| < vol(V_x(0))/4 or |V_y(1)| < vol(V_y(0))/4 then
    return failure
end if

```

Algorithm 1 Find directed path $x, \ldots, y$ (continued)

$t = 1$ ▷ Step 2
while $\mathrm{vol}(V_x(t)) < \frac{\mathrm{vol}(G)}{100}$ or $\mathrm{vol}(V_y(t)) < \frac{\mathrm{vol}(G)}{100}$ **do**
  $U(t+1) \leftarrow U(t) \setminus V_{min}(t)$
  $V_{min}(t+1) \leftarrow N(V_{min}(t)) \cap U(t+1)$
  $V_{max}(t+1) \leftarrow V_{max}(t)$
  if $V_{min}(t+1) \cap V_{max}(t+1) \neq \emptyset$ **then**
    return success
  end if
  if $\mathrm{vol}(V_{min}(t+1)) < 2\,\mathrm{vol}(V_{min}(t))$ **then**
    return failure
  end if
  $t = t + 1$
end while

if $N^+(V_x(t)) \cap V_y(t) \neq \emptyset$ **then** ▷ Step 3
  return success
else
  return failure
end if

3.3.2 Proof of Theorem 17

We prove Theorem 17 by bounding the probability of failure at each step in Algorithm 1.

Step 1: We claim that P(failure in Step 1) $\leq 2n^{-c/16}$.

Proof. Letting $X = |N^+(x)|$, from our minimum degree requirement $> c \log n$, we have that $E(X) \geq \frac{c}{2}\log n$. Using the lower tail of the Chernoff bound with $= E(X)/2$, we obtain:

$$P\left(X \leq \frac{c}{4}\log n\right) \leq n^{-c/16}.$$

The claim thus follows by the union bound. □

Step 2 We bound the probability of failure after one iteration of the while loop, given that failure has not yet occurred. We will require the following lemma:

Lemma 5. If $\mathrm{vol}(V_{min}(t)) = \varepsilon\,\mathrm{vol}(G)$, then $\mathrm{vol}(U(t+1)) \geq (1 - 4\varepsilon)\mathrm{vol}(G)$.

Proof. Equivalently stated, we must show

$$\mathrm{vol}(U(t+1)) \geq \mathrm{vol}(G) - 4\mathrm{vol}(V_{min}(t)):$$

To avoid failure , recall that we require $\mathrm{vol}(V_{min}(t+1)) \leq 2\mathrm{vol}(V_{min}(t))$. The volume of each successive neighborhood grows geometrically, thereby ensuring both

$$2\mathrm{vol}(V_{min}(t)) > \sum_{\{i \in [t] : V_{min}(i) = V_x(i)\}} \mathrm{vol}(V_x(i));$$
$$2\mathrm{vol}(V_{min}(t)) > \sum_{\{i \in [t] : V_{min}(i) = V_y(i)\}} \mathrm{vol}(V_y(i)):$$

So, at time t, we have removed a set with volume at most $4\mathrm{vol}(V_{min}(t))$ from $U(t+1)$. □

Using this lemma, we find a lower bound on the number of edges between $V_{min}(t)$ and the unexplored vertices $U(t+1)$ in terms of $\mathrm{vol}(V_{min}(t))$ using the strengthened discrepancy inequality. Observe

$$\begin{aligned}
e(V_{min}(t); U(t+1)) &\geq \frac{\mathrm{vol}(V_{min}(t))\mathrm{vol}(U(t+1))}{\mathrm{vol}(G)} - \frac{\sqrt{\mathrm{vol}(V_{min}(t))\mathrm{vol}(\overline{V_{min}(t)})\mathrm{vol}(U(t+1))\mathrm{vol}(\overline{U(t+1)})}}{\mathrm{vol}(G)} \\
&\geq (1-4\varepsilon)\mathrm{vol}(V_{min}(t)) - \frac{\sqrt{(1-\varepsilon)(1-4\varepsilon)\mathrm{vol}(G)^2 4\varepsilon^2\mathrm{vol}(G)^2}}{\mathrm{vol}(G)} \\
&= (1-4\varepsilon)\mathrm{vol}(V_{min}(t)) - 2\varepsilon\,\mathrm{vol}(V_{min}(t))\sqrt{(1-\varepsilon)(1-4\varepsilon)} \\
&= \left(1 - 4\varepsilon - 2\varepsilon\sqrt{(1-\varepsilon)(1-4\varepsilon)}\right)\mathrm{vol}(V_{min}(t))
\end{aligned}$$

$$> \frac{\mathrm{vol}(V_{\min}(t))}{2};$$

where the second inequality holds since, using the lemma along with the fact that $\varepsilon < 1/100$ yields $U(t+1) \leq \frac{24}{25}(\mathrm{vol}(G))$, implying $(1-4\varepsilon)4\varepsilon\,\mathrm{vol}(G)^2 \leq \mathrm{vol}(U(t+1))\mathrm{vol}(\overline{U(t+1)})$.

Next, considering our undirected host graph G, we partition $N(V_{\min}(t)) \setminus U(t+1)$ into two sets: one in which there are "many" edges from $V_{\min}(t)$ to $N(V_{\min}(t)) \setminus U(t+1)$ and one in which there are "few". More precisely, define

$$T_1 = \{v \in N(V_{\min}(t)) \setminus U(t+1) : e(v; V_{\min}(t)) \geq 5\log_2 n\};$$
$$T_2 = \{v \in N(V_{\min}(t)) \setminus U(t+1) : e(v; V_{\min}(t)) < 5\log_2 n\}.$$

Upon randomly orienting our graph, we have

$$P\left[v \notin \vec{N}(V_{\min}(t))\right] \leq \left(\frac{1}{2}\right)^{5\log_2 n} = n^{-5}.$$

And since there are at most n vertices in T_1, we have

$$P\left[\exists v \in T_1 : v \notin \vec{N}(V_{\min}(t))\right] \leq n^{-4}.$$

Observe that at least half of the edges leaving $V_{\min}(t)$ go to T_1 or at least half of the edges leaving $V_{\min}(t)$ go to T_2. We thus divide the remainder of the proof into two cases, showing that in either case, the volume of the next neighborhood of $V_{\min}(t)$ is sufficiently large to avoid incurring failure.

<u>Case 1</u> $e(V_{\min}(t); T_1) \geq \frac{1}{2}e(V_{\min}(t); U(t+1))$.

Combining our assumption for Case 1 along with the above bound on $e(V_{\min}(t); U(t+1))$ yields

$$\frac{\mathrm{vol}(V_{\min}(t))}{4} \leq e(V_{\min}(t); T_1).$$

Considering this fact, assume for sake of contradiction that $\mathrm{vol}(T_1) \leq 2\mathrm{vol}(V_{\min}(t))$. Then

$$e(V_{\min}(t); T_1) \leq \frac{\mathrm{vol}(V_{\min}(t))\mathrm{vol}(T_1)}{\mathrm{vol}(G)} + \sqrt{\mathrm{vol}(V_{\min}(t))\mathrm{vol}(T_1)}$$

$$\le \frac{vol(T_1)}{100} + \sqrt{vol(V_{min}(t))vol(T_1)}$$
$$\le \frac{vol(V_{min}(t))}{50} + \sqrt{2}\ vol(V_{min}(t));$$

which is a contradiction.

<u>Case 2</u> $e(V_{min}(t); T_2) \ge \frac{1}{2}e(V_{min}(t); U(t+1))$.

Again, we have

$$\frac{vol(V_{min}(t))}{4} \le e(V_{min}(t); T_2):$$

And by definition, each vertex in T_2 has fewer than $5\log_2 n$ edges from $V_{min}(t)$. Putting these two facts together yields

$$|T_2| \ge \frac{vol(V_{min}(t))}{20\log_2 n}:$$

Upon orienting edges randomly, we have

$$E(|V_{min}(t+1)|) \ge \frac{\log 2}{40}\frac{vol(V_{min}(t))}{\log n}:$$

Using the lower tail of the Chernoff bound with $= E(|V_{min}(t+1)|)/2$ and the fact that $vol(V_{min}(t)) \ge vol(V_{min}(1)) \ge \frac{c^2}{4}\log^2 n$ yields

$$P\left(|V_{min}(t+1)| \le \frac{\log 2}{80}\frac{vol(V_{min}(t))}{\log n}\right) \le e^{-(\frac{c^2\log 2}{1280})\log n} = n^{-\frac{c^2\log 2}{1280}}:$$

Thus, from our minimum degree requirement, we know that with probability $1 - n^{-\frac{c^2\log 2}{1280}}$, we have

$$vol(V_{min}(t+1)) \ge \frac{c\log 2}{80}vol(V_{min}(t)) > 2vol(V_{min}(t));$$

for $c > 160/\log 2$. Finally, we can bound the probability of failure in Step 2. The number of iterations are bounded above by $2\log_2(vol(G))$. Since Case 1 has a higher probability of failure, we have

$$P(\text{failure in Step 2}) \le 2\log_2(vol(G))n^{-4}:$$

Step 3 At this final stage, the successive neighborhoods of each vertex have grown sufficiently large (i.e. $\mathrm{vol}(V_x(t)) \geq \frac{\mathrm{vol}(G)}{100}$ and $\mathrm{vol}(V_x(t)) \geq \frac{\mathrm{vol}(G)}{100}$). We find a lower bound on the number of edges between them using the discrepancy inequality:

$$\begin{aligned} e(V_x(t), V_y(t)) &\geq \frac{\mathrm{vol}(V_x(t))\mathrm{vol}(V_y(t))}{\mathrm{vol}(G)} - \varepsilon\sqrt{\mathrm{vol}(V_x(t))\mathrm{vol}(V_y(t))} \\ &\geq \frac{1-100\varepsilon}{10000}\mathrm{vol}(G) \\ &\geq \frac{1-100\varepsilon}{10000} cn\log n, \end{aligned}$$

where the second inequality follows since the left hand-side is monotonically increasing in $\mathrm{vol}(V_x(t))\mathrm{vol}(V_y(t))$ for $\mathrm{vol}(V_x(t))\mathrm{vol}(V_y(t)) \geq (\mathrm{vol}(G)/100)^2$ given our choice of ε.

Here, failure occurs only if all these edges are oriented in the same direction. Thus,

$$P(\text{failure in Step 3}) \leq 2^{-\frac{1-100\varepsilon}{10000} cn\log n} = n^{-\frac{cn(1-100\varepsilon)}{\log_2(e)10{,}000}},$$

thereby completing the proof of Theorem 17.

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Chapter 4

Related future work

4.1 Maximum hitting time for directed graphs

Beyond what we've mentioned, there appears to be little known about how the principal ratio of a directed graph may be related to other important random walk parameters. In the undirected case, extremal families for the principal ratio have been shown to be extremal for other parameters, like maximum hitting time. In particular, the expected hitting time $H_G(u,v)$ between two vertices u and v in a graph G is the expected number of steps it takes to reach v, in a random walk starting at u. In the undirected case, Brightwell and Winkler proved the following:

Theorem 20 (Brightwell, Winkler [8]). Let G be a connected graph on vertices and $H_G(u,v)$ denote the expected hitting time it takes to reach v from u in a simple random walk on G. Then

$$\max_{G:|V|=n} \max_{u,v \in V} H_G(u,v) \qquad \frac{4}{27}n^3;$$

and is achieved by vertices u,v in a lollipop graph (see Figure 4.1), consisting of a clique of size $\frac{2n+1}{3}$ containing vertex u, to which a path on the remaining vertices ending in v has been attached.

Tait and Tobin showed that the extremal graphs for the principal ratio are also lollipop graphs, albeit with a slightly different clique size and path length [48].

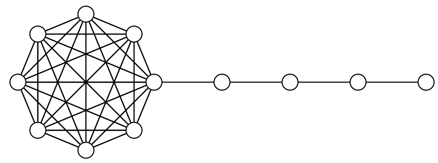

Figure 4.1 : The lollipop graph described in Theorem 20 for $n = 12$.

Indeed, lollipop graphs have been shown to be extremal for a number of other random walk parameters, such as cover time [23] and commute time [29].

In the directed case, it seems plausible that the extremal graphs for principal ratio are part of a larger family of directed graphs extremal for maximum hitting time. Below, we formulate this conjecture formally and derive an explicit formula for the maximum hitting time of our principal ratio extremal graphs. This shows that, whereas the maximum hitting time is on the order of n^3 in the undirected case, maximum hitting time is at least on the order of $(n-1)!$ in the directed case.

Formally, the hitting time between two vertices $u, v \in V(D)$ in a random walk on a directed graph D is denoted $H_D(u, v)$ and defined by

$$H_D(u;v) = \begin{cases} 1 + \dfrac{1}{d^+(u)} \displaystyle\sum_{w \in N^+(u)} H_D(w;v); & \text{if } u \neq v; \\ 0 & \text{if } u = v. \end{cases} \tag{4.1}$$

The maximum hitting time of a directed graph, denoted (D), is the maximum hitting time between all pairs of vertices, i.e.

$$(D) = \max_{u,v \in V(D)} H_D(u;v).$$

Question 1. What is

$$(n) = \max_{D : |V(D)| = n} (D);$$

and what is the family of directed graphs achieving this maximum?

Conjecture 1.

$$(n) = \lceil (e-1)(!n-1) - 3H_n + 2H_{n+1} \rceil;$$

where H_n denotes the n-th harmonic number and $!n$ is the left factorial defined as $!n = \sum_{i=0}^{n-1} i!$. Asymptotically,

$$\lim_{n \to \infty} (n) = (e-1)\,(n-1)!.$$

D_5

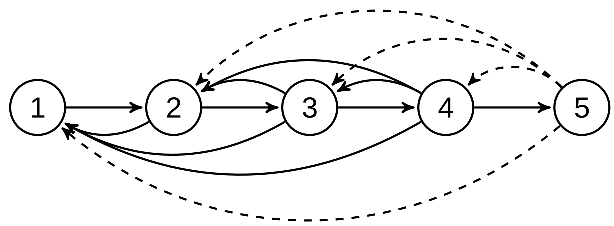

Figure 4.2 : An illustration of $\mathcal{D}_n$ for $n = 5$. A particular member of $\mathcal{D}_n$ contains all the undashed edges and any positive number of the dashed edges.

We conjecture $\tau(n)$ is maximized precisely by the family of n-vertex directed graphs $\mathcal{D}_n$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set

$$\{(v_i, v_{i+1}) : \text{for all } 1 \le i \le n-1)\} \cup \{(v_j, v_i) : \text{for all } 1 \le i < j \le n-1\} \cup S,$$

for any nonempty $S \subseteq \{(v_n, v_i) : i \in [n-1]\}$. See Figure 4.2 for an illustration.

The posited maximum hitting time in Conjecture 1 is the maximum hitting time of the principal ratio extremal graphs we found in Theorem 10, which belong to the family $\mathcal{D}_n$ defined above. Below, we find a closed-form expression for the maximum hitting time for $\mathcal{D}_n$ in terms of n.

Claim 2. Let $D \in \mathcal{D}_n$, where $\mathcal{D}_n$ is defined above. Then $\tau(D) = H_D(1, n) = \lceil (e-1)(n-1)! - 3H_n + 2H_{n+1} \rceil$. Asymptotically,

$$\lim_{n \to \infty} \tau(D) = (e-1) \cdot (n-1)!.$$

Proof. For ease of notation, let $x_{i,n} := H_D(i, n)$ for $i = 1, \ldots, n$. Applying the formula (4.1) for hitting time for D yields $x_{1,n} = 1 + x_{2,n}$ and $x_{n,n} = 0$ and for $i = 2, \ldots, n-1$,

$$x_{i,n} = 1 + \frac{1}{i}(x_{1,n} + \cdots + x_{i-1,n} + x_{i+1,n}).$$

We claim that

$$x_{i,n} = f_i + x_{i+1,n}, \tag{4.2}$$

where $f_k = \sum_{i=1}^{k} \frac{k!}{i!}$. We prove this by strong induction. Since $x_1 = 1 + x_2$ and $x_2 = 3 + x_3$, we have $f_1 = 1$ and $f_2 = 3$. Assume the claim holds for $i = 1, \ldots, k-1$. Then

$$
\begin{aligned}
x_{k;n} &= 1 + \frac{1}{k}(x_{1;n} + \cdots + x_{k-1;n} + x_{k+1;n}) \\
&= 1 + \frac{1}{k}\left((f_1 + x_{2;n}) + (f_2 + x_{3;n}) + \cdots + (f_{k-1} + x_{k;n}) + x_{k+1;n}\right) \\
&= 1 + \frac{1}{k}\Big((f_1 + \cdots + f_{k-1}) + (f_2 + \cdots + f_{k-1}) + \cdots + \\
&\qquad (f_{k-1}) + (k-1)x_k + x_{k+1;n}\Big) \\
&= 1 + \frac{1}{k}\left(\sum_{i=1}^{k-1} i f_i + (k-1)x_{k;n} + x_{k+1;n}\right) \\
&= \left(k + \sum_{i=1}^{k-1} i f_i\right) + x_{k+1;n};
\end{aligned}
$$

where $f_k = k + \sum_{i=1}^{k-1} i f_i$. Next, we show that $k + \sum_{i=1}^{k-1} i f_i = \sum_{i=1}^{k} \frac{k!}{i!}$ with $f_1 = 1$. It is easy to verify the base case holds for $k = 2$; assume the result holds for $k = 1, \ldots, s-1$. Then

$$
\begin{aligned}
f_s &= s + \sum_{i=1}^{s-1} i f_i \\
&= 1 + (s-1) + \sum_{i=1}^{(s-1)-1} i f_i + (s-1) f_{s-1} \\
&= 1 + f_{s-1} + (s-1) f_{s-1} \\
&= 1 + s f_{s-1} \\
&= 1 + s \sum_{i=1}^{s-1} \frac{(s-1)!}{i!} = 1 + \sum_{i=1}^{s-1} \frac{s!}{i!} = \sum_{i=1}^{s} \frac{s!}{i!}.
\end{aligned}
$$

Rewriting $x_{1;n}$ using (4.2) and using the fact that $x_{n;n} = 0$, we have that $x_{1;t} = \sum_{k=1}^{n-1} f_k$. So, substituting the above formula for f_i into this equation yields

$$
x_{1;n} = \sum_{k=1}^{n-1} \sum_{i=1}^{k} \frac{k!}{i!}. \tag{4.3}
$$

First, we claim that $\max(D) = x_{1;n}$, that is, $x_{1;n} \geq x_{i;j}$ for all $i, j \in [n]$. By (4.2), for any given n, $x_{i;n} > x_{i+1;n}$ and then hence $x_{1;n} \geq x_{i;n}$ for any $i \in [n]$. By (4.3), it is clear that $x_{1;n} \geq x_{1;j}$ for any $j \in [n]$. Hence $\max(D) = x_{1;n}$.

Next we shall show

$$x_{1,n} = \lceil (e-1)(\omega n - 1) - 3H_n + 2H_{n+1} \rceil . \tag{4.4}$$

We claim that

$$f_k = \lfloor (e-1) \cdot k! \rfloor , \tag{4.5}$$

where $\lfloor x \rfloor$ denotes the largest integer no larger than x. Since f_k is an integer and

$$e - 1 = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i!} ,$$

to prove (4.5) it suffices to show

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{k!}{i!} - f_k < 1 . \tag{4.6}$$

We start from the left-hand side of the inequality (4.6), for $k \geq 1$,

$$\begin{aligned}
& \lim_{n\to\infty} \sum_{i=1}^{n} \frac{k!}{i!} - f_k \\
= {} & \lim_{n\to\infty} \sum_{i=1}^{n} \frac{k!}{i!} - \sum_{i=1}^{k} \frac{k!}{i!} \\
= {} & \lim_{n\to\infty} \sum_{i=k+1}^{n} \frac{k!}{i!} \\
= {} & \frac{1}{k+1} + \frac{1}{(k+2)(k+1)} + \lim_{n\to\infty} \sum_{i=k+3}^{n} \frac{k!}{i!} \\
= {} & \frac{k+3}{(k+2)(k+1)} \\
& + \frac{1}{(k+2)(k+1)} \cdot \lim_{n\to\infty} \left(\frac{1}{k+3} + \frac{1}{(k+4)(k+3)} + \cdots + \frac{1}{n \cdots (k+4)(k+3)} \right) \\
< {} & \frac{k+3}{(k+2)(k+1)} + \frac{1}{(k+2)(k+1)} \cdot \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots + \frac{1}{2^{n-k-2}} \right) \\
= {} & \frac{k+4}{(k+2)(k+1)} \\
\leq {} & \frac{5}{6} < 1 ,
\end{aligned}$$

which proves (4.5). For convenience, we denote the difference between $(e-1) \cdot k!$ and f_k by Δ_k. From the proof above, we can see that

$$\Delta_k := (e-1) \cdot k! - f_k > \frac{k+3}{(k+2)(k+1)} = \frac{2}{k+1} - \frac{1}{k+2} ,$$

and

$$\epsilon_k < \frac{k+4}{(k+2)(k+1)} = \frac{3}{k+1} - \frac{2}{k+2}.$$

For $n \geq 1$, we see that

$$2H_n - H_{n+1} - \frac{1}{2} < \sum_{k=1}^{n-1} \epsilon_k < 3H_n - 2H_{n+1}, \tag{4.7}$$

where H_n is the n-th Harmonic number,

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4},$$

and $\gamma \approx 0.5772156649$ is the Euler–Mascheroni constant. This observation is straightforward because

$$2\frac{1}{k+1} - \frac{1}{k+2} < \epsilon_k < 3\frac{1}{k+1} - 2\frac{1}{k+2}.$$

Now recalling that

$$x_{1;n} = \sum_{k=1}^{n-1}\sum_{i=1}^{k} \frac{k!}{i!} = \sum_{k=1}^{n-1} f_k,$$

we have

$$x_{1;n} = \sum_{k=1}^{n-1} \left((e-1)\cdot k! - \epsilon_k\right). \tag{4.8}$$

Next we claim that $x_{1;n}$ is the only integer in the open interval

$$\left((e-1)(!n-1) - 3H_n + 2H_{n+1},\ (e-1)(!n-1) - 2H_n + H_{n+1} + \frac{1}{2}\right),$$

where $!n$ is left factorial defined as

$$!n := \sum_{i=0}^{n-1} i!.$$

Combining (4.7) and (4.8), it is apparent to see that $x_{1;n}$ lies in the open interval above. We also need check the length of the interval which is equal to

$$-2H_n + H_{n+1} + \frac{1}{2} - (-3H_n + 2H_{n+1}) = \frac{1}{2} + H_n - H_{n+1} = \frac{1}{2} - \frac{1}{n},$$

which implies the length of the interval is bounded by $1/2$. It means $x_{1;n}$ is the only integer in this interval and then hence

$$x_{1;n} = \lceil (e-1)(!n-1) - 3H_n + 2H_{n+1} \rceil.$$

Next we shall show the asymptotics of $x_{1;n}$. It suffices to show

$$\lim_{n\to\infty} \frac{x_{1;n}}{(e-1)\cdot(n-1)!} = 1;$$

which is equivalent to

$$\lim_{n\to\infty} \frac{(e-1)\omega(n-1) - 3H_n + 2H_{n+1}}{(e-1)\cdot(n-1)!} = 1.$$

Clearly,

$$\lim_{n\to\infty} \frac{H_n}{(e-1)\cdot n!} = 0;$$

and it is easy to see that

$$\lim_{n\to\infty} \frac{\omega(n+1)}{n!} = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{x_{1;n}}{(e-1)\cdot(n-1)!} = 1.$$

□

4.2 Upper bounds on λ_1 for directed graphs

In Section 24, we established lower bounds for λ_1, the first non-trivial eigenvalue of the normalized Laplacian, extending an analogous result for undirected graphs to the directed case. For the related question of upper bounds on λ_1, we note that the obvious bound,

$$\lambda_1 \le \frac{n}{n-1}; \tag{4.9}$$

extends trivially to the directed case, as the trace of the directed normalized Laplacian $\mathcal{L}$ is still n and $\lambda_0 = 0$. Furthermore, equality holds if $G = \vec{K}_n$, the complete directed graph on n vertices. However, whereas for undirected graphs only the complete graph K_n has $\lambda_1 > 1$ (see [14, Lemma 1.7]), directed graphs other than $\vec{K}_n$, may have $\lambda_1 > 1$. For example, deleting any edge of $\vec{K}_3$ yields a graph with $\lambda_1 \approx 1.19$; in general, we leave it as an exercise to show that for any $e \in E(\vec{K}_n)$, we have $\lambda_1(\vec{K}_n - e) > 1$ for all $n \ge 3$.

One can obtain more refined upper bounds on λ_1 by using the variational characterization of λ_1 and cleverly constructing an eigenfunction. For example, Nilli proved the following:

Theorem 21 ([41, 14]). Let G be a graph with diameter $D \geq 4$ and maximum degree k. Then

$$\lambda_1 \geq 1 - 2\frac{\sqrt{k-1}}{k}\left(1 - \frac{2}{D}\right) + \frac{2}{D}.$$

In the directed case, such bounds for the first nontrivial eigenvalue of the normalized Laplacian are not known. It would be of interest to obtain more nuanced bounds than (4.9) on λ_1, in terms of directed diameter and other parameters.

4.3 Hamiltonian cycles in random orientations

In Chapter 3, we showed that under a mild minimum degree condition and mild eigenvalue condition, all orientations of a given graph are strongly connected. Furthermore, we showed that these conditions are essentially best possible. Nonetheless, our result could be improved by showing that these (or similar) conditions guarantee a graph satisfies (with high probability) properties stronger than strong connectedness, like Hamiltonicity. Namely, for a given graph G, what minimal conditions on its minimum degree and spectral gap of its normalized Laplacian guarantee that a random orientation has a directed Hamiltonian cycle, with high probability?

It seems unlikely that one could adapt our proofs here to guarantee Hamiltonicity, given that our approach was fundamentally geared towards proving strong connectedness. Grötschel and Haray [26] showed that the only graphs for which every strong orientation is Hamiltonian are the complete graph and cycle graph. Extending work of Krivelevich and Sudakov [30] for regular graphs, Butler and Chung [12] showed that an eigenvalue condition involving the spectral gap of the combinatorial Laplacian implies an "almost regular" graph is Hamiltonian. It is unclear whether such results have any bearing on proving existence of Hamiltonian cycles in random orientations under analogous spectral conditions. A key tool utilized, the so-called rotation-extension technique due to Pósa [44], does not appear to have an analog in the directed case.

4.4 Other extremal problems on the principal ratio

Here, we brie y review extremal problems related to Theorem 10, where we proved a sharp upper bound on the principal ratio of a directed graph. Note that the three constructions in the statement of Theorem 10 achieving the maximum principal ratio have maximum in-degree and out-degree equal to 1. Additionally, these constructions are also dense, having at least $\binom{n}{2}$ edges out of the $2\binom{n}{2}$ edges possible. One natural problem would be to determine the maximum of the principal ratio when in-degree or out-degree are bounded, or when the number of edges is not very large. Here are several ways to formulate such questions:

Question 2. For given $n; k; j$ with $k; j < n$, what is the maximum principal ratio over all simple strongly connected directed graphs on n vertices with maximum out-degree at most k and maximum in-degree at most j? That is, determine $(n; k; j)$ where

$$(n;k;j) = \max_{\substack{D : |V(D)| = n \\ d^{+}_{max} = k;\ d_{max} = j}} (D):$$

Question 3. For given $n; m$, what is the maximum principal ratio over all strongly connected directed graphs on vertices with at most m edges? That is, determine $(n; m)$ where

$$(n;m) = \max_{\substack{D : |V(D)| = n \\ |E(D)| \quad m}} (D):$$

For both of the above questions, it would be of interest to characterize the extremal family of graphs achieving the maximum.

In random walks on unweighted directed graphs, the probability of moving from a vertex to any of its neighbors is equally likely. For the general case of random walks on weighted directed graphs, the probability of moving from vertex u to v is proportional to edge weight w_{uv}. In this case, unless edge weights are bounded, the principal ratio can be made arbitrarily large by making the weights of in-edges incident to a particular vertex arbitrarily small. The following question is of interest:

Question 4. What is the maximum principal ratio over all strongly connected weighted directed graphs on n vertices with edge weight function $w : E \to \mathbb{R}^+ \cup \{0\}$ having minimum value $w_{min} > 0$ and maximum value $w_{max} \leq 1$?

We remark that some of the techniques and constructions used in Chapter 2 may be useful when considering the weighted case. For example, consider an edge-weighting of the construction D_1 defined in Theorem 10 with $w_{v_i v_{i+1}} = w_{min}$ for $1 \leq i \leq n-1$ and $w_{uv} = 1$ for all other edges $(u, v) \in E(D_1)$. By adapting a greedy argument similar to that used in Proposition 6, it is not too difficult to show that for w_{min} sufficiently small, this weighted digraph has principal ratio at least $(w_{max}/w_{min})^{n-2}(n-2)!$. This serves as a lower bound for the maximum principal ratio in the weighted case.

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