



Spectral Threshold for Extremal Cyclic Edge-Connectivity

Sinan G. Aksoy¹ · Mark Kempton² · Stephen J. Young³

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Abstract

The universal cyclic edge-connectivity of a graph G is the least k such that there exists a set of k edges whose removal disconnects G into components where every component contains a cycle. We show that for graphs of minimum degree at least 3 and girth g at least 4, the universal cyclic edge-connectivity is bounded above by $(\Delta - 2)g$ where Δ is the maximum degree. We then prove that if the second eigenvalue of the adjacency matrix of a d -regular graph of girth $g \geq 4$ is sufficiently small, then the universal cyclic edge-connectivity is $(d - 2)g$, providing a spectral condition for when this upper bound on universal cyclic edge-connectivity is tight.

Keywords Edge connectivity · Cyclic edge connectivity · Graph spectrum

1 Introduction

The traditional notion of graph edge-connectivity is the smallest k such that there exists a set of edges $S \subseteq E(G)$ with $|S| = k$, where $G \setminus S$ is disconnected. Note that traditional edge-connectivity does not stipulate any conditions on properties of the components of $G \setminus S$. The notion of *conditional edge-connectivity*, introduced by

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✉ Sinan G. Aksoy
sinan.aksoy@pnnl.gov

Mark Kempton
mkempton@mathematics.byu.edu

Stephen J. Young
stephen.young@pnnl.gov

¹ Pacific Northwest National Laboratory, Seattle, WA 98109, USA

² Mathematics Department, Brigham Young University, Provo, UT 84602, USA

³ Pacific Northwest National Laboratory, Richland, WA 99354, USA

Harary [15], extends the traditional edge-connectivity by stipulating that components of $G \setminus S$ satisfy some given property. More precisely:

Definition 1 (*Harary* 1983 [15]) Let P be any property of a graph $G = (V, E)$, and let $S \subset E(G)$. The *universal P -connectivity* is the minimum $|S|$ such that $G \setminus S$ is disconnected, and every component of $G \setminus S$ has property P .

We note there are several formulations of conditional connectivity; for instance, the qualifier *universal* reflects that *every* component of $G \setminus S$ has P , whereas *existential* conditional connectivity relaxes this condition to *some* component satisfying P . Harary’s introduction of conditional connectivity focused on surveying a wide swath of possible properties, including planarity, degree and diameter restrictions, and the property of being cyclic, Hamiltonian, or Eulerian. Arguing that “the most fruitful topics are those suggested by applications”, Harary aimed to provide a framework for devising connectivity concepts that are meaningful in applications. In particular, Harary references the pertinence of conditional connectivity to the analysis of computer network reliability [4], and VLSI and separator problems [18], among others. For instance, he notes that guarantees on connected component sizes reflect the resilience of computer networks to disruptions.

In this work, we consider the universal P -edge-connectivity, where P is the property of containing a cycle. We call this the universal cyclic edge-connectivity of G , which we denote by $\kappa'_c(G)$. Prior to Harary’s work, Bollobas alludes to universal cyclic edge-connectivity [5, p. 113], and Harary proved the universal cyclic edge-connectivity of the balanced complete bipartite graph $K_{n,n}$ is $n^2 - 2n$ for n even. The cycle condition is also natural for a number of applications, such as network reliability, as the existence of a cycle is necessary to guarantee multiple paths between pairs of vertices. Cyclic edge-connectivity has also been utilized as a condition to solve other problems, such as in integer flow conjectures [12], and early attempts [28] at proving the four color theorem. Lastly, cyclic-edge connectivity has also garnered interest due to its close relationships with other types of connectivity [16, 25] and so-called n -extendable graphs [26].

Before proceeding, we note universal cyclic edge-connectivity is equal to several other notions of connectivity. Instead of requiring *every* component of $G \setminus S$ have a cycle, a number of researchers (cf. [17, 21, 22]) define cyclic edge-connectivity has the smallest edge cut set S such that *at least 2* components possess a cycle. Defined in this way, while a cyclic edge cut need not be a universal cyclic edge cut, it is nonetheless straightforward¹ to show this notion of cyclic-edge connectivity is equal to universal cyclic edge-connectivity. Despite this equivalence, we still utilize the universal formulation of cyclic edge-connectivity because restricting the permissible cuts affords advantages in our proof techniques. Lastly, we note other notions of edge-connectivity are equivalent to cyclic edge-connectivity for certain

¹ Suppose that S is a minimal cyclic edge cut that is not universal. Then $G \setminus S$ has a tree component T as well as two components C_1 and C_2 that contain a cycle. Now there is some edge $e \in S$ which is incident to T . Consider the edge cut $S' = S \setminus e$. Since e is not incident between C_1 and C_2 , $G \setminus S'$ still has two distinct components, C'_1 and C'_2 which contain C_1 and C_2 as subgraphs, respectively. But then S' is a cyclic edge cut, a contradiction.

parameter settings, or for certain families of graphs. For instance, Latifi et al. [16] propose a new measure of conditional connectivity for large multiprocessor systems, requiring every vertex have degree k in $G \setminus S$. As observed in [29], when $k = 2$ and the minimum degree is at least 3, this quantity is equal to cyclic edge-connectivity. For more on the relationship between cyclic edge-connectivity and other connectivity measures, see [25].

We take a spectral approach for studying universal cyclic edge-connectivity. We prove that for a d -regular graph with girth g , with $d \geq 5$, a bound on the second eigenvalue of the adjacency matrix is sufficient to guarantee that $\kappa'_o(G) = (d - 2)g$, which is the largest possible universal cyclic edge-connectivity (see Theorems 1 and 2 below). Furthermore, we construct a family of d -regular graphs that show our spectral condition is necessary. Prior work has established a number of spectral bounds for traditional edge and vertex connectivity; see [1, 7, 13] and the references contained therein. Furthermore, researchers have also investigated connectivity and conditional connectivity with restrictions on component sizes for many families of graphs [8, 9, 11, 19, 30]. However, spectral bounds on conditional connectivity appear far more rare. One exception, however, is recent work by Zhang [30], who makes use of spectral tools to determine the cyclic edge-connectivity of strongly regular graphs. Despite the pervasive use of spectral methods as a proof technique, Zhang's approach does not immediately lead to an eigenvalue condition for cyclic edge-connectivity. Focusing on the more general case of d -regular graphs, our bound establishes a spectral threshold for extremal cyclic edge-connectivity, depending on girth and degree. We note all strongly-regular graphs which are at least 5-regular satisfy the condition of our theorem. It is interesting to note, that all 4-regular strongly regular graphs also satisfy the stated spectral threshold even though not satisfying our degree condition.

2 Main Tools and Notation

For a graph $G = (V, E)$, let $\kappa'_o(G)$ denote the cyclic edge-connectivity of G . For vertex subsets $X, Y \subseteq V$, let $E(X, Y)$ denote the set of edges between X and Y , and let $e(X, Y) := |E(X, Y)|$. Further, let $G[X]$ denote the subgraph induced by X . One of our primary tools will be the following well-known lemma from [24].

Lemma 1 (Lemma 3.1 of [24]) *Let G be a graph of order n , $X \subseteq V$, and λ_2 the second largest eigenvalue of the adjacency matrix. Then*

$$e(X, \bar{X}) \geq (d - \lambda_2) \frac{|X|(n - |X|)}{n}.$$

The other key ingredient of the proof is the following theorem of Alon, Hoory, and Linial which provides a lower bound for the number of vertices in a graph with a given average degree and girth. This result may be thought of as an irregular generalization of the result of Moore (see [3, p. 180]), which lower bounds the number of vertices in a d -regular graph of a given diameter.

Irregular Moore Bound (Alon et al. [2]) *The number of vertices n in a graph of girth g and average degree at least $d \geq 2$ satisfies $n \geq n_0(d, g)$ where*

$$n_0(d, g) = \begin{cases} 1 + d \sum_{i=0}^{r-1} (d-1)^i & \text{if } g = 2r + 1 \\ 2 \sum_{i=0}^{r-1} (d-1)^i & \text{if } g = 2r \end{cases} .$$

To demonstrate the applicability of these tools, we first obtain the following naïve spectral bound on conditional edge-connectivity in terms of girth when conditioned on the size of the minimal component.

Proposition 1 *Let γ be the size (number of edges) of the smallest edge-cut of a d -regular graph with second largest adjacency eigenvalue λ_2 which results in components of size at least k , and let the girth be g . Then*

$$\gamma \geq k(d - \lambda_2) \left(1 - \frac{k}{n_0(d, g)} \right),$$

where $n_0(d, g)$ is as in the Irregular Moore Bound.

Proof We first note that by the minimality of the cut, there is some set X such that $\gamma = e(X, \bar{X})$. Thus, by Lemma 1, we have that

$$\gamma = e(X, \bar{X}) \geq (d - \lambda_2) \frac{|X|(n - |X|)}{n} .$$

Applying the aforementioned Irregular Moore Bound of Alon et al. [2] immediately yields the desired result. □

The form of this lower bound is quite close to the trivial upper bound on γ , $k(d - 2) + 2$, stemming from the case where $G[X]$ is a k -vertex tree. In some sense, this extremal example makes universal cyclic edge-connectivity the natural strengthening of the size-limited connectivity.

The final tool we will utilize to study the universal cyclic edge-connectivity of d -regular graphs is the *ear decomposition* of a graph, which as stated in [6], may be defined as follows.

Ear Decomposition. *For a subgraph F of G , an ear of F in G is a nontrivial path in G whose ends lie in F but whose internal vertices do not. An ear decomposition of a 2-edge-connected graph G is a nested sequence (G_0, G_1, \dots, G_k) of subgraphs of G such that*

- (i) G_0 is a cycle,
- (ii) $G_{i+1} = G_i \cup P_i$, where P_i is an ear of G_i in G , for $0 \leq i \leq k$,
- (iii) $G_k = G$.

3 Cyclic Edge-Connectivity

Before proceeding with our main result, we first address the existence of and upper bounds on cyclic-edge connectivity. Indeed, for acyclic or unicyclic graphs, it is clear the cyclic edge connectivity does not exist. The work of Lovász [23] and Dirac [10] provide a complete characterization of the class of graphs with no pairs of vertex disjoint cycles. Roughly speaking, these are graphs obtained from K_5 , a wheel, and $K_{3,t}$ plus any subset of edges connecting vertices in the three element class, and a forest plus a dominating vertex by the duplication and subdivision of edges and the addition of trees. However, as no assurances are provided on the relative sizes of these cycles, these works do not yield an upper bound on the universal cyclic edge-connectivity even in the d -regular case. Lou and Holton [20, Lemma 1] provide a straight-forward argument that for d -regular girth g graphs, the cyclic edge connectivity is bounded above by $(d - 2)g$. However, their approach does not appear to generalize to the irregular graphs. To address this, we provide in Lemma 2 explicit conditions for the existence of a girth-length cycle which induces a universal cyclic edge cut.

Lemma 2 *Let G be a graph with minimum degree $d \geq 3$ and girth $g \geq 4$. If G is not $K_{3,t}$, then there exists a cycle C of length g such that every component of $G - E(C, \overline{C})$ contains a cycle.*

As an immediate corollary we have the following.

Theorem 1 *Let G be a graph with minimum degree at least 3 and girth ≥ 4 . If G is not $K_{3,t}$, then $\kappa'_o(G) \leq (\Delta - 2)g$ where Δ is the maximum degree of G .*

Proof of Lemma 2 We first consider the case that $g \geq 5$ and let C be a cycle of length g . As C is an induced cycle it is clear that $G[C]$ contains a cycle. Now consider an arbitrary component $G[X]$ of $G - E(C, \overline{C})$ and let $x \in X$. As $g \geq 5$, the vertex x has at most one neighbor in C as otherwise there exists a cycle of length at most $\lfloor \frac{g}{2} \rfloor + 1 < g$. But then the minimum degree of $G[X]$ is 2 and hence it contains a cycle.

Now suppose that G is a graph with girth 4 and is not $K_{3,t}$. Let x, y be vertices of G such that the set of common neighbors, $Z = \{z_1, \dots, z_k\}$ has size at least 2. The existence of such a pair of points is guaranteed as the girth is 4 and there is an induced 4-cycle in G . It is worth mentioning that, since the girth of G is 4, none of the vertices adjacent to a vertex in Z are adjacent to either x or y . We now consider the component structure of $G - \{x, y\}$. Let Z_1, \dots, Z_{k_x} be the vertex sets for components that contain an element of Z and let X_1, \dots, X_{k_x} , (respectively Y_1, \dots, Y_{k_y}) be the vertex sets for the components such that $E(X_i, \{x\}) \neq \emptyset$ and $X_i \cap Z = \emptyset$ (respectively, $E(Y_i, \{y\}) \neq \emptyset$ and $Y_i \cap Z = \emptyset$). We note that the collection of vertex sets $\{X_1, \dots, X_{k_x}\}$ and $\{Y_1, \dots, Y_{k_y}\}$ are not necessarily distinct, however, this potential duplicate naming will not affect our subsequent analysis. We will show the desired cycle is in $\{x, y\} \cup \bigcup_i Z_i$. We note the components induced by any X_i or Y_j do not provide an obstruction. This is easy to see as by definition any

vertex in X_i or Y_j is incident to precisely one of $\{x, y\}$ as otherwise it would belong to Z . Thus $G[X_i]$ and $G[Y_j]$ have minimum degree at least 2, and thus, contain a cycle. As a consequence, we may restrict our attention to the components $G[Z_1], \dots, G[Z_{k_c}]$ without loss of generality.

We first consider the case where one of the components, say $T = G[Z_1]$, is a tree. As the minimum degree of G is 3 and Z contains every vertex adjacent to both x and y , the leaves of T are given by $Z_1 \cap Z$. Let z, z' be two vertices of maximum distance in T and let the unique path between them be given by $z = t_0, t_1, \dots, t_\ell, t_{\ell+1} = z'$. We first consider the case where $\ell \geq 2$ and so t_1 and t_ℓ are distinct vertices. We note that $E(\{x, y\}, \{t_1, t_\ell\}) = \emptyset$ as both t_1 and t_ℓ are incident to elements of Z . Thus t_1 and t_ℓ have degree at least 3 in T and thus there are vertices t'_1 and t'_ℓ that are incident to t_1 and t_ℓ , respectively. By the maximality of the distance between z and z' in T and the uniqueness of the shortest path in trees, t'_1 and t'_ℓ are also leaves and hence in Z . But then consider the components of $G - C$, where C is the cycle $\{y, z, t_1, t'_1\}$. Note that $\{x, z', t_\ell, t'_\ell\}$ is a cycle and disjoint from $\{y, z, t_1, t'_1\}$ and thus is present in $G - C$. Furthermore, the components of $T - \{z, t_1, t'_1\}$ as well as the components $G[Z_2], \dots, G[Z_{k_c}]$ are all incident to x and thus form a single component which contains the cycle $\{x, z', t_\ell, t'_\ell\}$. Hence, C is the desired cycle.

Thus we may now assume that any tree component among $G[Z_1], \dots, G[Z_{k_c}]$ has diameter 2 and a unique vertex not in Z . Suppose $G[Z_1]$ is such a component and let v be the unique element of $G[Z_1]$ not in Z . Since v is adjacent to an element of Z , we have that v is not adjacent to $\{x, y\}$, has degree at least 3, and $\{x, y, v\} \cup (Z_1 \cap Z)$ induces copy of $K_{3,t}$ in G . As G is not equal to $K_{3,t}$, this implies that one of Z_2, X_1 , or Y_1 exists and is not empty. Suppose first that X_1 exists and let $z, z' \in Z_1 \cap Z$. Consider the components of $G - \{y, z, v, z'\}$. As every element of $Z - \{z, z'\}$ is incident to x , $G[X_1]$ contains a cycle, and there is an edge between X_1 and x , we have that every component formed contains a cycle. A similar argument holds if Y_1 exists by exchanging the roles of x and y . Finally, assume that no components of the type X_i or Y_j exist, but Z_2 exists. Let $z, z' \in Z \cap Z_1$ and consider the components of $G - \{y, z, v, z'\}$. As all the components Z_3, \dots, Z_{k_c} as well as the vertices of $Z_1 - \{z, z'\}$ are connected to x , in order to show that the all the components of $G - \{y, z, v, z'\}$ have a cycle it suffices to show that $G[Z_2 \cup \{x\}]$ contains a cycle. To that end, if $|Z_2 \cap Z| = 1$, then every vertex in $Z_2 - Z$ is adjacent to at most one vertex in $\{x, y\} \cup Z$ and hence $G[Z_2 - Z]$ has minimum degree 2 and a cycle. Otherwise, $|Z_2 \cap Z| \geq 2$ and $G[Z_2 \cup \{x\}]$ contains a cycle by taking a path between distinct elements of Z in $G[Z_2]$ joined by the vertex x .

At this point, we may assume without loss of generality that $G[Z_i]$ is not a tree for all i . Suppose that the induced graph $G[\cup_i Z_i]$ has at least two vertices of degree 1, z, z' . Then as z and z' are on no cycles in $G[\cup_i Z_i]$, the components of $G[\cup_i Z_i - \{z, z'\}]$ all contain cycles. This gives that $\{x, z, y, z'\}$ is the desired cycle C . Thus we may assume that there is at most 1 vertex of degree 1 in $G[\cup_i Z_i]$, which we call z , and let $z' \in Z - \{z\}$. As the induced subgraph $G[\cup_i Z_i - \{z, z'\}]$ has at most one vertex of degree 1 (potentially the unique common neighbor of z and z'), all of its components contain a cycle and again $\{x, z, y, z'\}$ is the desired cycle. We may now

assume that for i , the induced subgraph $G[Z_i]$ is not a tree and every element of Z has degree at least 2 in the relevant component. Suppose that $k_z \geq 2$ and let $z \in Z_1 \cap Z$, $z' \in Z_2 \cap Z$. Every vertex in $Z_1 - Z$ is adjacent to at most one of $\{x, y, z, z'\}$ and so has degree at least 2 in $G[Z_1 - \{z, z'\}]$, while the elements of $Z \cap Z_1 - \{z, z'\}$ are incident to neither of z or z' , and thus also have minimum degree 2. This implies that $G[Z_i - \{z, z'\}]$ has minimum degree 2 for all i and thus contains a cycle, and hence $\{x, z, y, z'\}$ is the desired cycle.

Finally, we may now assume that there is a single component Z_1 of $G - \{x, y\}$ that contains all elements of Z and further, that component is not a tree and every element of $Z \cap Z_1$ has degree at least two in the component. Now fix two elements $z, z' \in Z$ and let F be the forest of tree components of $G[Z_1 - \{z, z'\}]$. Clearly if F is empty, then $\{x, z, y, z'\}$ is the desired cycle. Thus we may assume that F is non-empty and let L be the leaves of F . As Z is an independent set and every vertex of z has degree at least 2 in $G[Z_1]$, we note that $L \cap Z = \emptyset$. Further, as the minimum degree is at least 3, any $\ell \in L$ is adjacent to at least two of $\{x, y, z, z'\}$. As $\ell \notin Z$, it can not be adjacent to both x and y . Additionally, ℓ can not be adjacent to one of $\{x, y\}$ and one of $\{z, z'\}$ as this forms a triangle. Thus every element of L is adjacent to both z and z' . Now suppose that there is some tree $T \in F$ such that $E(T, \{x, y\}) = \emptyset$ and let ℓ, ℓ' be leaves of that tree. But then, z and z' are antipodal points in a 4-cycle such that their common neighbors are in distinct components of $G - \{z, z'\}$. Specifically, ℓ, ℓ' are in a different component than $\{x, y\}$ and thus by previous arguments the desired cycle exists. Thus we may assume that every component of F is adjacent to either x or y . But then, if $|Z| \geq 4$, we have that for any two leaves $\ell, \ell' \in L$, the cycle $\{z, \ell, z', \ell'\}$ is the desired cycle. Specifically, if $\bar{z}, \bar{z}' \in Z - \{z, z'\}$ then the tree components of $G - \{x, y, z, z', \ell, \ell'\}$ are all connected to the cycle $\{x, \bar{z}, y, \bar{z}'\}$ via either x or y . To complete the proof we note that the common neighbors of z and z' include $\{x, y\}$ and L , and thus have at least 4 elements. Thus, by repeating the arguments above with $\{z, z'\}$ in the role of $\{x, y\}$, we may assume that $|Z| \geq 4$ as needed. \square

One might hope Lemma 2 could be extended to girth 3 graphs with a similarly small set of exceptions as the girth 4 case. However, it is relatively easy to identify infinite families of counterexamples from the work of Lovász [23] and Dirac [10]. For example, the wheel graph on (see Fig. 1) on any number of vertices forms a counterexample as every 3-cycle in the graph involves the central hub as well as an edge from the from the outer cycle. Thus the removal of a 3-cycle destroys every cycle in the graph. Another infinite family of counterexamples can be formed by taking $K_{3,t}$ and adding a non-empty set of edges to the partition of size 3. In this case, every 3-cycle uses 2 of the vertices of partition of size 3 and thus because the graph is bipartite there are not enough vertices remaining on that side to form a cycle (see Fig. 1).

In fact, by adding relatively few vertices it is possible to transform an arbitrary triangle-free graph into a counterexample to Lemma 2 where the universal cyclic edge-connectivity exists. Specifically, let G be an arbitrary triangle-free graph and let T be a tree with at least 3 leaves and minimum non-leaf degree 3. Adding two adjacent vertices c, c' which are connected to all the leaves of T and an arbitrary

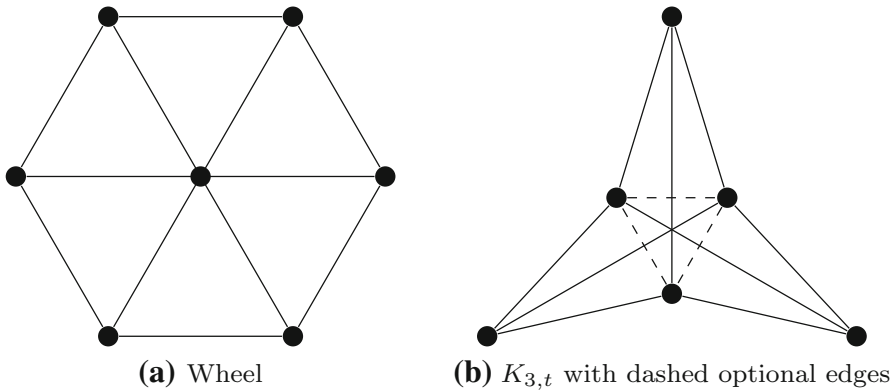


Fig. 1 Examples from two infinite families of girth 3 graphs where the universal cyclic edge-connectivity does not exist

independent set S in G , results in a graph G' in which every triangle uses the edge $\{c, c'\}$ and a vertex in S or a leaf of T . Thus, deleting the edges incident to any triangle in G' results a cycle-free component, namely T or T with a single leaf removed. However, with the added restriction that the graph is d -regular, the only counterexamples the authors are aware of are K_4 and K_5 . Thus it is possible there is a finite set of exceptions for a d -regular, girth 3, version of Lemma 2.

Theorem 2 *Let G be a d -regular graph with $d \geq 5$ and second largest adjacency eigenvalue λ_2 . If G has girth $g \geq 4$ and*

$$d - \lambda_2 \geq \frac{2(d - 2)g}{n_0(d - \frac{2}{r-1}, g)},$$

where $r = \lfloor g/2 \rfloor$ and n_0 is as in the Irregular Moore Bound, then $\kappa'_o(G) = (d - 2)g$. If G has girth $g = 3$, the cyclic-edge connectivity exists, and

$$d - \lambda_2 \geq 6 - \frac{12}{d},$$

then $\kappa'_o(G) = 3d - 6$.

Proof Let S be a minimal set of edges such that $G - S$ is disconnected with all components containing a cycle and that $|S| < (d - 2)g$. By minimality, we may assume that there is some set of vertices X such that $S = E(X, \bar{X})$, $|X| \leq |\bar{X}|$, and $H = G[X]$ contains a cycle. Further, we may assume that X is the minimal cardinality set yielding an edge cut of size $|S|$. Now, by Lemma 1, we have that

$$e(X, \bar{X}) \geq (d - \lambda_2) \frac{|X||\bar{X}|}{n} \geq (d - \lambda_2) \frac{|X|}{2}.$$

Thus, to prove the desired result for $g \geq 4$, by Lemma 2 (or Lemma 1 in [20]) it suffices to show that

$$|X| \geq n_0 \left(d - \frac{2}{r-1}, g \right) \geq \frac{2(d-2)g}{d-\lambda_2}.$$

We first observe that if H is a simple cycle, then $|X| \geq g$ and $e(X, \bar{X}) \geq (d-2)g$, as desired. Thus assume without loss of generality that H is not a cycle. Further note that, if $x \in X$ is such that there is some cycle $C \in G[X - \{x\}]$, then the degree of x in H is at least $\lceil \frac{d}{2} \rceil$ as otherwise $G[X - \{x\}]$ contains the cycle C , $G[\bar{X} \cup \{x\}]$ contains the same cycle as $G[\bar{X}]$, and $e(X - \{x\}, \bar{X} \cup \{x\}) \leq e(X, \bar{X})$. As a consequence, the minimum degree in H is at least 2.

At this point it is possible to observe that H is 2-edge-connected. Specifically, suppose that there exists two disjoint sets X_1 and X_2 such that $X_1 \cup X_2 = X$ and $e(X_1, X_2) \leq 1$. As the minimum degree in H is at least 2, both $G[X_1]$ and $G[X_2]$ contain some cycle. By minimality of the edge cut and that $X_1, X_2 \subset X$, we have that $e(X_1, \bar{X}_1), e(X_2, \bar{X}_2) \geq e(X, \bar{X}) + 1$, thus

$$e(X, \bar{X}) = e(X_1, \bar{X}_1) + e(X_2, \bar{X}_2) - 2 \geq 2e(X, \bar{X}),$$

a contradiction.

Since H is 2-edge-connected, there exists an ear decomposition for H [27]. Specifically, there exists a cycle $C \in H$ as well as paths P_1, \dots, P_k such that the internal vertices of P_i are disjoint from $C \cup \left(\bigcup_{j=1}^{i-1} P_j \right)$ and $H = C \cup \left(\bigcup_{j=1}^k P_j \right)$. Since we may assume that H is not a cycle, we have that there is a non-zero number of paths in the ear decomposition. So we may consider the last path in the ear decomposition, P_k . By construction of the ear decomposition, any internal vertex in P_k will have degree 2 in H but will be disjoint from the cycle C . But as such vertices have degree at least $\lceil \frac{d}{2} \rceil$ and $d \geq 5$, no such vertex exists and P_k is a single edge $e = \{x, y\}$. By the properties of the ear decomposition $H - e$ is a 2-edge-connected graph and hence there are at least two, edge-disjoint, paths between x and y in $H - e$, denote them by Q and Q' . Without loss of generality we assume that the total length of Q and Q' is minimized. Now suppose there exists vertices a and b that are on both Q and Q' , but in opposite orders. That is, $Q = xQ_1aQ_2bQ_3y$ and $Q' = xQ'_1bQ'_2aQ'_3y$. We can then construct two new walks from x to y , $xQ_1aQ'_3y$ and xQ'_1bQ_2y which have total shorter length. Thus, if $x = a_0, a_1, \dots, a_{t-1}, a_t = y$ are the intersection points of Q and Q' , they occur in the same order in both Q and Q' . As a consequence, $Q \cup Q'$ can be thought of as a series of vertex incident cycles C_0, C_1, \dots, C_t such that $a_j, a_{j+1} \in C_j$ and $C_{i-1} \cap C_i = \{a_i\}$. Now for every vertex v in H (except potentially x, y if $t = 1$ and a_1 if $t = 2$), there is some cycle not containing v and hence the degree of v is at least $\lceil \frac{d}{2} \rceil \geq 3$. As the degree of x and y are at least 2 and the degree of a_1 is at least 4, this implies that average degree of H is at least $2 + \epsilon$ for some $\epsilon > 0$.

As H has average degree $2 + \epsilon$ and girth at least g , by the Irregular Moore Bound, we have that $|X| \geq n_0(2 + \epsilon, g)$. But then, since G is d -regular and $e(X, \bar{X}) < (d-2)g$, we have the average degree is at least

$$d - \frac{(d-2)g}{n_0(2+\epsilon, g)}.$$

In particular, we have that ϵ must satisfy that

$$(d-2-\epsilon)n_0(2+\epsilon, g) < (d-2)g.$$

Consider first the case where $g = 2r \geq 4$, and note that

$$\begin{aligned} (d-2-\epsilon)n_0(2+\epsilon, 2r) &= (d-2-\epsilon) \left(2 \sum_{j=0}^{r-1} (1+\epsilon)^j \right) \\ &= 2(d-2-\epsilon) \frac{(1+\epsilon)^r - 1}{\epsilon} \\ &\geq 2(d-2-\epsilon) \left(r + \binom{r}{2} \epsilon \right). \end{aligned}$$

Thus we need to have

$$2(d-2-\epsilon) \left(r + \binom{r}{2} \epsilon \right) < 2r(d-2),$$

which can be rearranged to

$$\binom{r}{2} \epsilon \left(d-2 - \frac{2}{r-1} - \epsilon \right) < 0.$$

As we already have that $\epsilon > 0$, this implies that $\epsilon > d-2 - \frac{2}{r-1}$ and thus $|X| \geq n_0(d - \frac{2}{r-1}, 2r)$.

Finally consider the case where $g = 2r + 1 \geq 5$. In this case we have that

$$\begin{aligned} (d-2-\epsilon)n_0(2+\epsilon, 2r+1) &= (d-2-\epsilon) \left(1 + (2+\epsilon) \sum_{j=0}^{r-1} (1+\epsilon)^j \right) \\ &\geq (d-2-\epsilon) \left(1 + (2+\epsilon) \sum_{j=0}^{r-1} (1+j\epsilon) \right) \\ &= (d-2-\epsilon) \left(1 + (2+\epsilon) \left(r + \binom{r}{2} \epsilon \right) \right) \\ &= (d-2-\epsilon) \left(1 + 2r + \left(2 \binom{r}{2} + r \right) \epsilon + \binom{r}{2} \epsilon^2 \right) \\ &= (d-2-\epsilon) \left(1 + 2r + r^2 \epsilon + \binom{r}{2} \epsilon^2 \right) \end{aligned}$$

Thus, we have that ϵ satisfies that

$$\epsilon \left(((d-2)r^2 - g) + \left((d-2) \binom{r}{2} - r^2 \right) \epsilon - \binom{r}{2} \epsilon^2 \right) < 0.$$

Letting

$$f(\epsilon) = - \binom{r}{2} \epsilon^2 + \left((d-2) \binom{r}{2} - r^2 \right) \epsilon + (d-2)r^2 - g,$$

it is easy to see that

$$\begin{aligned} \lim_{\epsilon \rightarrow -\infty} f(\epsilon) &= -\infty, \\ f(0) &= (d-2)r^2 - 2r - 1 > 0, \\ f(d-2) &= -g, \quad \text{and} \\ \lim_{\epsilon \rightarrow \infty} f(\epsilon) &= -\infty \end{aligned}$$

Thus $f(\epsilon)$ has one root in $(-\infty, 0)$ and one in $(0, d-2)$. Let ϵ^* be the root of $f(\epsilon)$ in $(0, d-2)$, then we have that $\epsilon \geq \epsilon^*$ and $|X| \geq n_0(2 + \epsilon^*, 2r + 1)$. Observing that,

$$\begin{aligned} f\left(d-2 - \frac{2}{r-1}\right) &= - \binom{r}{2} \left(d-2 - \frac{2}{r-1}\right)^2 \\ &\quad + \left((d-2) \binom{r}{2} - r^2 \right) \left(d-2 - \frac{2}{r-1}\right) + (d-2)r^2 - g \\ &= \binom{r}{2} \left(- \left(d-2 - \frac{2}{r-1}\right)^2 + \left(d-2 - \frac{2}{r-1}\right) \left(d-2 - \frac{2r}{r-1}\right) \right. \\ &\quad \left. + \frac{2(d-2)r}{r-1} - \frac{g}{\binom{r}{2}} \right) \\ &= \binom{r}{2} \left(\left(d-2 - \frac{2}{r-1}\right) \frac{2-2r}{r-1} + \frac{2(d-2)r}{r-1} - \frac{g}{\binom{r}{2}} \right) \\ &= \binom{r}{2} \left(\frac{2(d-2)}{r-1} + \frac{4r-4}{(r-1)^2} - \frac{4r+2}{r(r-1)} \right) \\ &= \frac{2(d-2)(r-1)r + (4r-4)r - (4r+2)(r-1)}{2(r-1)} \\ &= \frac{2(d-2)(r-1)r + 2 - 2r}{2(r-1)} \\ &= (d-2)r - 1 \\ &> 0 \end{aligned}$$

Thus $\epsilon^* > d-2 - \frac{2}{r-1}$ and $|X| > n_0\left(d - \frac{2}{r-1}, 2r + 1\right)$.

Finally, the case for $g = 3$ proceeds similarly as above except that $n_0(2 + \epsilon, 3) =$

$3 + \epsilon$ and thus ϵ^* can be determined to be exactly $d - 5$, yielding $\kappa'_o(G) \geq 3d - 6$. The matching upper bound on $\kappa'_o(G)$ for girth 3 follows immediately by applying Lemma 1 from [20]. \square

Given the relatively weak spectral condition required in Theorem 2 and the extensive use of the Irregular Moore Bound in the proof, one might naturally wonder whether any spectral condition is required at all. Here we briefly provide a family of d -regular examples (for $d \geq 5$) showing that, for girth 4 at least, the spectral condition is required.

Example 1 We first note that for a d -regular graph with girth $g = 4$,

$$n_0\left(d - \frac{2}{r - 1}, g\right) = n_0(d - 2, 4) = 2(1 + (d - 3)) = 2d - 4,$$

and thus if $\lambda_2 \leq d - \frac{4d-8}{d-2} = d - 4$ then the cyclic edge connectivity is $4d - 8$. In contrast to this, our family of examples has second largest eigenvalue at least $d - \frac{5-\sqrt{17}}{2} > d - 4$ and has cyclic edge connectivity $2d - 6$.

We begin our construction with the odd case. In this case the graph can be partitioned into 4 sets,

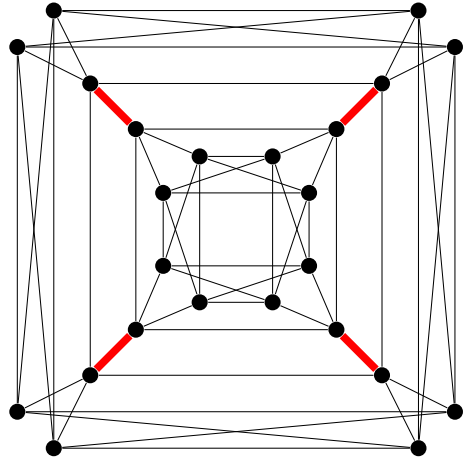
$$\begin{aligned} A &= \{a_1, \dots, a_{4d-12}\} \\ B &= \{b_1, \dots, b_{2d-6}\} \\ C &= \{c_1, \dots, c_{2d-6}\} \\ D &= \{d_1, \dots, d_{4d-12}\}. \end{aligned}$$

Two vertices $a_i, a_j \in A$ are adjacent if i and j differ (cyclically) by one of $1, 3, \dots, d - 2$. A similar relation holds for pairs of vertices in D . Two vertices b_i and b_j are adjacent if i and j differ (cyclically) by one of $1, 3, \dots, d - 4$. All remaining edges are between A and B , B and C , or C and D . Specifically, b_i is adjacent to a_i, a_{i+d-3} , and c_i . A analogous relationship holds for vertices in C . It easy to verify that this graph has degree d and girth 4. Furthermore, the edge cut between B and C has size $d - 1$ with both components containing many cycles. The $d = 5$ case is depicted in Fig. 2.

In the case that d is even, we first generate the graph for degree $d + 1$ and then remove a perfect matching contained in each of the sets A, B, C , and D . The existence of such a perfect matching is easily observed by noting that each set contains a even-length cycle through all the vertices.

Finally we turn to the second largest eigenvalue of each of these graphs. We note that in both the odd and even cases the sets A, B, C , and D form an equitable partition of the graph with associated equitable partition matrix

Fig. 2 The 5-regular graph from Example 1. The edge cut of size 4 is bolded and highlighted in red



$$\mathcal{E} = \begin{bmatrix} d-1 & 1 & 0 & 0 \\ 2 & d-3 & 1 & 0 \\ 0 & 1 & d-3 & 2 \\ 0 & 0 & 1 & d-1 \end{bmatrix}.$$

The spectrum of \mathcal{E} is $\left\{d, d - \frac{5 \pm \sqrt{17}}{2}, d - 3\right\}$. Since this is an equitable partition of the graph, the spectrum of the graph includes $d - \frac{5 - \sqrt{17}}{2}$. Furthermore, as the eigenvalues not associated with \mathcal{E} are orthogonal to the indicator vector for each part in the partition (see for instance Section 9.3 of [14]), the largest eigenvalue not induced by \mathcal{E} is bounded above by the maximum eigenvalue of

$$\begin{bmatrix} \lambda_2(A) & 1 & 0 & 0 \\ 2 & \lambda_2(B) & 1 & 0 \\ 0 & 1 & \lambda_2(C) & 2 \\ 0 & 0 & 1 & \lambda_2(D) \end{bmatrix},$$

where $A, B, C,$ and D are the subgraphs induced by the respective sets.

In the case that d is odd, $A, B, C,$ and D are all Abelian Cayley graphs over a cyclic group and so it straightforward to derive that

$$\lambda_2(A) = \lambda_2(D) = \sum_{j=1}^{\frac{d-1}{2}} 2 \cos\left(\frac{(2j-1)\pi}{2d-6}\right) = \frac{\sin\left(\frac{(d-1)\pi}{2d-6}\right)}{\sin\left(\frac{\pi}{2d-6}\right)} < d-1$$

$$\lambda_2(B) = \lambda_2(C) = \sum_{j=1}^{\frac{d-3}{2}} 2 \cos\left(\frac{(2j-1)\pi}{d-3}\right) = 0.$$

The largest eigenvalue of

$$\begin{bmatrix} d-1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & d-1 \end{bmatrix}$$

is $\frac{\sqrt{d^2-4d+12}+d}{2}$ which is equal to $d - \frac{5-\sqrt{17}}{2}$ when $d = 5$ and strictly smaller for larger values of d .

As the example for even d results from removing a repeated matching, the Courant-Weyl inequalities gives that $\lambda_2(B), \lambda_2(C) \leq 1$ while maintaining that $\lambda_2(A), \lambda_2(D) < d - 2$. Thus, a similar argument as above yields that for all even $d \geq 4$, the second eigenvalue of every graph in the family is $d - \frac{5-\sqrt{17}}{2}$.

We note that while the above example does not show that Theorem 2 is tight, it does however show that (at least for girth 4) the bound is of the correct order.

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Declarations

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